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A COMPENDIOUS
COURSE OF MATHEMATICS,

Theoretical and Practical,

FOR THE USE OF SCHOOLS, AND OF THOSE WHO HAVE
NOT THE ADVANTAGE OF A MASTER.

I. ALGEBRA.
II. PLANE TRIGONOMETRY.

BY

J. R. YOUNG,

LATE PROFESSOR OF MATHEMATICS IN BELFAST COLLEGE; AND AUTHOR OF
"WALKINGHAME'S TUTOR'S ASSISTANT, IMPROVED."

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PREFACE.

It is intended that the present volume shall be followed by two others, each, like this first, complete in itself, but so connected with the other two, that the whole may form a systematic compendium of modern elementary mathematics. Whether or not this intention be fully carried out, will depend upon the manner in which the portion of the plan now submitted to the public may be received.

Should this be such as to encourage the prosecution of the original design to its completion, the three volumes will embrace an amount of mathematical instruction sufficient to meet the demand of most seminaries of education.

The volumes in contemplation will contain, in addition to the more advanced parts of pure mathematics, the applications of these to mechanical philosophy, optics, and physical astronomy.

The author has not space at his disposal here to give any summary of the particulars introduced into the following pages; he believes, however, that they may be consulted, not altogether without profit, even by those who are already in some degree familiar with Algebra and Trigonometry.

Such persons may, for instance, gather some small additions to their knowledge from the remarks, generally inclosed in brackets, about series—combinations—the incommensurability of the circular circumference, &c. And perhaps one or two things, common to all books of the kind—such as the theory of the greatest common measure, and the investigation of the fundamental formulæ of Analytical Trigonometry—may be found to be placed in a more agreeable and satisfactory light. But for the reason just assigned, these matters cannot be enlarged upon here: and the author can only add that, from the pains taken with the manuscript, and with the proofs, he believes that very few errors indeed have escaped his notice.

June, 1855.

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A COMPENDIOUS COURSE OF MATHEMATICS.

I.—ALGEBRA.

Preliminary Notions of the Subject.

1. The rules and operations of Algebra are only those of Arithmetic applied to letters as well as to figures. You will gain a notion of how *letters* can be introduced into calculation as well as figures, by reflecting upon what is done in working a common addition or subtraction example in the arithmetic of compound (or *concrete*) quantities; as for instance, in adding together several sums of money, or in subtracting one sum of money from another.

ADDITION.

26*l*	3*s*	2*d*
17*l*	1*s*	3*d*
43*l*	2*s*	4*d*
9*l*	5*s*	1*d*
	7*s*	
95*l*	18*s*	10*d*

Thus, the operation in the margin, though one of common arithmetic, differs but very little from addition of algebra; and, as far as the part performed by the letters is concerned, it does not differ at all.

The letters *l, s, d,* in this example, are *symbols,* standing for *pounds, shillings,* and *pence.* It is universally agreed that, in money operations, these symbols shall have the meanings here stated, and no other meanings; yet it is plain that if any person chose to do so, he might employ these same symbols *l, s, d,* in his own private practice, to stand for *days, hours,* and *minutes;* or indeed for any other quantities.

The result of the operation above—taking the symbols in the commonly received sense—is 95 pounds 18 shillings and 10 pence; but had these symbols stood for days, hours, and minutes, as just supposed, then the result would have been interpreted 95 days 18 hours and 10 minutes.

In working the above example, as in common arithmetic, the symbols *l, s, d* would not be written against each of the several items as here; they would of course be placed over them once for all: they have been repeated in this place solely for the purpose of giving an algebraical form to the example, and having gone over it, you may consider that you have worked an exercise in addition of algebra. It is true that if this exercise were given among algebraical examples, the vacant spaces which separate the distinct quantities from each other would be filled up: the quantities would be linked together by a mark or *sign*, called *the sign of addition;* it is an upright cross +, and is read "*plus;*" it implies that the quantity to which it is prefixed goes to the increase of what precedes. In the money example, as given at first, this is sufficiently understood, but the sign + expressly declares it. Introducing therefore the sign of addition, as in the margin, we have a perfect specimen of the algebraical form; the result being correct, whatever *l*, *s*, *d* may stand for.

$$\begin{array}{r@{\,}r@{\,}r}
26l+ & 3s+ & 2d \\
17l+ & 1s+ & 3d \\
43l+ & 2s+ & 4d \\
9l+ & 5s+ & 1d \\
 & 7s & \\
\hline
95l+ & 18s+ & 10d \\
\hline
\end{array}$$

If, instead of *l*, *s*, *d*, the symbols had been any other letters, *a*, *b*, *c*, for instance, or *x*, *y*, *z*, then, whatever quantities *a*, *b*, *c*, or *x*, *y*, *z* represent, you could find no difficulty in working the following examples in addition of algebra, provided you understand what has just been said.

$$\begin{array}{r@{\,}r@{\,}r}
26a+ & 3b+ & 2c \\
17a+ & 1b+ & 3c \\
43a+ & 2b+ & 4c \\
9a+ & 5b+ & 1c \\
 & 7b & \\
\hline
95a+ & 18b+ & 10c \\
\hline
\end{array}
\qquad
\begin{array}{r@{\,}r@{\,}r}
13a+ & 4b+ & 6c \\
21a+ & 7b+ & 3c \\
 & 2b+ & 5c \\
6a+ & 3b+ & 2c \\
 & & 9c \\
\hline
40a+ & 16b+ & 25c \\
\hline
\end{array}
\qquad
\begin{array}{r@{\,}r@{\,}r}
4x+ & 5y+ & 2z \\
18x+ & 12y+ & 10z \\
5x+ & 8y+ & 3z \\
7x+ & 11y+ & 14z \\
2x & + & 5z \\
\hline
36x+ & 36y+ & 44z \\
\hline
\end{array}$$

In examples of this kind you will find it the most convenient course always to begin with the first column on the left, not with the first on the right, as in arithmetic. In this latter science you add up the columns from right to left, for the sake of the carryings; but there are no carryings from one column to another in addition of algebra; you must see that such is the case, because the letters are not confined to any particular class of quantities, as pounds, shillings, and pence, or yards, feet, and inches, &c., but may stand for any quantities whatever. And it is plain that the addition may always be performed as above, though the quantities represented by the letters be unknown to us; it is not necessary that the meaning of *l*, *s*, *d* should previously be made known to us before we could perform the operation in the margin at page 1; for whatever *l* may stand for, 95*l* must represent the sum of the entire column of quantities under which the 95*l* is placed. In like manner, 18*s* represents the sum of the quantities in the second column, and 10*d* the sum of those in the third column, whatever *s* and *d* may stand for. It is of importance that you keep this truth constantly in mind, namely, that the interpretation of the *letters* is not at all necessary to the performance of the *operations;* it is with operations only that you will have to do at present, and the rules for the performance of these will now be given. When you can work with algebraical symbols with the same facility that you can work with the figures of common arithmetic, you will be shown how to apply algebra to special purposes—to the solution of particular questions, where a distinct meaning is given to the letters, just as a distinct meaning is given to *l*, *s*, *d*, and such-like symbols, in concrete arithmetic; but at present you will have to attend to operations only—not to interpretations.

DEFINITIONS.

2. The following are the signs of operation which are of most frequent use in algebra; there are a few others which will be explained hereafter.

1. $+$, *plus*, the sign for addition.

2. $-$, *minus*, the sign for subtraction. Thus, $a+b$ signifies that the quantity represented by a is to be *increased* by the quantity represented by b; and $a-b$ signifies that the quantity represented by a is to be *diminished* by that represented by b. If, for instance, a stand for the number 6, and b for the number 4, then $a+b$ would stand for 10, and $a-b$ for 2; that is, a plus b would be 10, and a minus b would be 2; in like manner $9+3$ is 12, and $9-3$ is 6.

3. $\times$, the sign for multiplication, implying that the numbers between which it is placed are to be multiplied together: thus, 6×4 implies that 6 and 4 are to be multiplied together, so that 6×4 is 24. In like manner $x \times 4$ or $4 \times x$ signifies that the quantity x is to be multiplied by 4; in other words, 4 times x is to be understood. But instead of the oblique cross for multiplication, a dot, placed between the factors, is also used for the same operation: thus, 6.4 equally implies the product of the two factors 6 and 4; and when only one of the factors is a number, and the other factors letters, the intervening sign for multiplication is usually suppressed altogether, the factors being merely written side by side, the numerical factor—when there is one—being always placed first: thus, $4x$ means four times x, equally with $4 \times x$ or $4.x$: and in like manner abx, $a \times b \times x$, $a.b.x$ equally mean the product of the three factors a, b, x, or of whatever these represent.

Although, as just stated, it is sufficient that algebraic quantities be written side by side, without any cross or dot between them, when the multiplication of those quantities—that is, of the numbers they stand for—is to be indicated, yet you will perceive that this plan will not do when the quantities are not algebraic symbols, but the arithmetical numbers themselves. If 2, 3, and 5 are to be multiplied together, the product would not be indicated by writing the factors thus, 235; for everybody would take 235 to mean two hundred and thirty-five: a sign, in cases of this kind, must always be placed between the numerical factors to indicate their multiplication: thus, the product just mentioned would be indicated either by $2 \times 3 \times 5$, or by $2.3.5$. But, in using this latter form, you must take care that each dot ranges with the *lower* part of the figures, not with the *upper*; because a dot before the upper part of a figure denotes that that figure is a decimal; thus ·3 means three-tenths.

4. $\div$, *divided by*, is the sign for division: thus $6 \div 2$ means 6 divided by 2; and $a \div b$ means a divided by b: that is, the quantity represented by a is to be divided by that represented by b.

There is another way of indicating division; namely, by writing the dividend above, and the divisor below a line of separation drawn between them; thus, $\frac{6}{2}$ is the same as $6 \div 2$; and $\frac{a}{b}$ the same as $a \div b$.

You will of course observe that the office of each of these signs of operation is simply to *indicate* the operation, not to *perform* it.

The sign $=$ is used to express *equality;* it is not a sign of *operation*, but a sign of *relation;* implying that the quantity written on the left of it is equal to that on the right; thus, $6+2=8$ is a brief way of stating that 6 increased by 2 is equal to 8; and $6-2=4$ affirms that 6 diminished by 2 is equal to 4; we should read these statements thus, "6 plus 2 equals 8," and "6 minus 2 equals 4." In like manner $a+b=c-d$ affirms that a plus b is equal to c minus d, understanding of course by a, b, &c., the quantities, whatever they be, which these letters are made to stand for.

The following are some examples on the preceding definitions.

(1) $13+4-6=11$. (2) $13a+4a-6a=11a$. (3) $13x+4x-6x=11x$.
(4) $5b-6b+3b=2b$. (5) $7y-3y-4y=0$. (6) $-4ax+3ax+6ax=5ax$.

In this 6th example, a minus quantity, that is, a subtractive quantity, is placed first, but in a set of quantities of which some are to be added and the others to be subtracted, the order in which they are placed before the eye is obviously of no moment; we have only to take care that the sum of those which are subtractive is actually deducted from the sum of those which are additive. When the leading quantity is plus or additive it is not necessary to prefix the plus sign, the absence of the minus suffices to show that the quantity is additive. It may be as well to mention here that additive and subtractive quantities are also called *positive* and *negative* quantities.

(7) $-5by+2by-3by+8by=2by$ (8) $3xyz-4xyz-2xyz=-3xyz$

Here the sum of the subtractive quantities exceeds the additive or positive quantity by $3xyz$; therefore the minus sign must be prefixed to the result to indicate this fact.

5. The figure or number prefixed to a quantity, as a multiplier of that quantity, is called its *coefficient;* thus, in *Ex.* 7 the coefficient of the first by is -5, that of the second 2, that of the third -3, and that of the fourth 8. The coefficient of $1ax$ is 1; but to express *once* ax, it is sufficient to write simply ax, which cannot be mistaken for anything else; unit-coefficients may therefore always be omitted; they are understood, though not expressed; thus the coefficient of x, or of yz, or of xyz, &c., is understood to be 1. Avoid the mistake of saying, in such a case, that the coefficient is *nothing*, for this would be an important error.

6. Any quantity written in the algebraic characters is called an *expression:* it is an expression of one *term* or a *simple* quantity when it is not separated into parts by the intervention of any plus or minus sign; thus each of the following is an expression of one term, that is, a simple quantity, or a simple expression :—

$2ax$, $5abxy$, $-6abcz$, $14pxy$, &c.

7. An expression consisting of two or more terms is called a *compound* expression, or a compound quantity. The following are compound expressions :—

$2ax+3by$, $3bxy-2az$, $ax+by-3cz$, &c.

The first and second of these consist each of two terms, the third of three terms: we should read this third expression thus: *a, x* plus *b, y* minus 3 times *c, z*; remembering that when we say *a, x*; *b, y*; &c., we mean the product of the factors which these letters represent.

The foregoing particulars being understood, you may proceed at once to Addition of Algebra, for which indeed the examples at page 2 have in some measure prepared you; by turning to them again you will perceive that, notwithstanding the presence of the *letters* in the several expressions added, the only actual work is with the *coefficients;* the column of coefficients is in each case added up, and then the proper letter written beside the sum.

ADDITION.

3. CASE I. *When the quantities to be added are all* LIKE *quantities, or quantities differing in nothing but in coefficients.*

RULE 1. Find the sum of the positive coefficients.

2. Find the sum of the negative coefficients.

3. To the *difference* of these two sums prefix the sign belonging to the *greater* sum; and the correct coefficient will be obtained.

4. Write beside this coefficient the *letters* common to all the quantities whose coefficients have been thus added, and the amount of the whole will be expressed.

[NOTE.*—It is obvious from these directions, that what is called *addition* of Algebra is not limited (as in arithmetic) to the finding the sum of a set of *additive* quantities, but extends to the finding the *aggregate* or *balance* of a set of quantities, of which some are additive and some subtractive; this balance is of course plus or minus, according as the sum of the additive quantities exceeds or falls short of the sum of the subtractive quantities. If a person were to prefix the sign minus to his items of expenditure, those of income being plus, he would at any time learn the state of his affairs by proceeding as above; the algebraical addition of the several sums would show the amount of the balance, the prefixed sign indicating whether it was in his favour or in that of his creditors.]

(1)	(2)	(3)	(4)
$2x$	$5ax+1$	$7axy-3b$	$8cz+2x-4$
$7x$	$3ax-2$	$-4axy-2b$	$-3cz-7x$
$-3x$	$6ax+4$	$-axy+b$	$5cz-4x+1$
$4x$	$-ax+3$	$8axy+6b$	$-9cz+3x-8$
$8x$	$-7ax-5$	$14axy-5b$	$6x+2$
$-2x$	$4ax+3$	$-3axy-4b$	-7
$16x$	$10ax+4$	$21axy-7b$	$cz \quad -16$

It is evident in all these examples, that the figures or coefficients are the only things that are actually added up, the letters being entire disregarded till the balance of the coefficients is obtained, when t letters, common to all the terms in the column, are copied down again it. Observe that the first column on the left is always that which *first* added, not the first on the right, as in arithmetic.

---

* Remarks enclosed in brackets *may* be omitted; but the reader is recommended *to attend to them,* as nothing superfluous is given.

Although you have nothing to do with the letters, beyond copyin them down against the sum of the coefficients, yet keep in mind th the results of the addition are true, whatever interpretation you give the letters; thus, in the fourth example, if we suppose $c = 4$, and $z =$ then the numerical value of all the quantities added will be $4 \times 6 - 1$ that is, $24 - 16$, or 8; so in example 3, if $a = 2$, $b = 3$, $x = 4$, an $y = 5$, then the amount of all the expressions, when thus interprete will be $21 \times 2 \times 4 \times 5 - 7 \times 3$, that is, $840 - 21$, or 819.

Examples for Exercise.

(1)

$$\begin{array}{r} 2ax-b \\ 3ax+4b \\ 7ax+2b \\ -4ax+5b \\ ax-3b \\ \hline \\ \hline \end{array}$$

(2)

$$\begin{array}{r} 4acx-3m \\ 6acx-2m \\ 5acx-\ m \\ -14acx+8m \\ 10acx+5m \\ \hline \\ \hline \end{array}$$

(3)

$$\begin{array}{r} 6xyz+4ab \\ 3xyz-7ab \\ -\ xyz-\ ab \\ -2xyz+3ab \\ 5xyz-\ ab \\ \hline \\ \hline \end{array}$$

(4)

$$\begin{array}{r} 7amz-3ny \\ -2amz+\ ny \\ -4amz-2ny \\ -6amz+4ny \\ 5amz+5ny \\ \hline \\ \hline \end{array}$$

(5)

$$\begin{array}{r} 4x-3y+7z \\ -2x-5y-8z \\ 6x+4y \\ 8y+3z \\ -7y-\ z \\ \hline \\ \hline \end{array}$$

(6)

$$\begin{array}{r} 3ax+4by-5cz \\ 5ax-3cz \\ -2ax-\ by+2cz \\ -7ax+2by-4cz \\ 8ax-5by \\ \hline \\ \hline \end{array}$$

(7)

$$\begin{array}{r} 4mz-2nx+17 \\ 3mz-5nx-\ 8 \\ -7mz-3nx-\ 1 \\ mz+\ nx+\ 4 \\ -8mz-4nx-\ 3 \\ \hline \\ \hline \end{array}$$

(8)

$$\begin{array}{r} 4ax-7by+3cz-4 \\ -5ax-3by-6cz+2\tfrac{1}{3} \\ -6ax+8by+\ cz-5 \\ 9ax+5by-2cz+7\tfrac{1}{3} \\ ax-4by+4cz-3 \\ \hline \\ \hline \end{array}$$

(9)

$$\begin{array}{r} 3\tfrac{1}{2}b+2c-5d+1 \\ -2\tfrac{1}{4}b-\tfrac{1}{2}c+3d-\tfrac{3}{4} \\ 6\tfrac{1}{4}b-\tfrac{3}{4}c+\tfrac{1}{2}d-7 \\ -2\tfrac{1}{2}b+\tfrac{1}{4}c-\tfrac{1}{4}d+\tfrac{1}{3} \\ b+\ c+\ d-4 \\ \hline \\ \hline \end{array}$$

In the foregoing examples all the *like* quantities, that is, all thos which involve exactly the same *letters*, are arranged one under anoth in a vertical column, and are thus put in proper order for the additio of the coefficients. But algebraic expressions to be added togeth seldom fall into this order; and you will, in general, have to arrang them for yourself, as in these examples:—

Add together the following expressions.

(10) $5ax-7by+2$, $3ax+2by-3$, $-5by+4$, $4ax-7$.

(11) $-4mz+3x$, $6mz-7x+1$, $-3mz-x-5$, $2x+3$.

(12) $\frac{1}{2}axy-2bz$, $\frac{3}{4}axy-\frac{1}{2}bz$, $\frac{1}{8}axy+\frac{3}{4}bz$, $5axy+bz$.

4. **Case II.** *When the quantities to be added are not all* LIKE *quantities.*

When a set of algebraical quantities intended to be added together are placed one under another, as in the first nine of the foregoing examples, but yet no attention paid to the orderly disposal of the like quantities in vertical columns, you will have to proceed in one or other of the two following ways:—1st. You may alter the arrangement so as to remove this want of uniformity, and then proceed with the addition as in the former case; thus, if the expressions be placed one under another at random, as in the first of the following groups, you may change the order of the terms, as in the second way of writing them, and then add up the columns.

$$\begin{array}{rcl}
3x-5y+2z & = & 3x-5y+2z \\
-4z+3x-y & = & 3x-y-4z \\
2y-z+a & = & \quad 2y-z+a \\
3a+5z-2x & = & -2x \quad +5z+3a \\
\hline
 & & 4x-4y+2z+4a, \text{ the sum.} \\
\hline
\end{array}$$

Or, 2nd. Without thus changing the order of the terms, you may proceed as follows, which is the common rule for this second case of addition.

Rule 1. Commencing with the quantity at the top of the first column, pick out all the quantities that are its *like*, wherever you find them, and write their sum under the first column.

2. In like manner, passing to the quantity at the top of the second column, add to it all the quantities that are its like, and so on.

3. If there are any terms among the expressions that have no like at top of any column, add these by themselves and connect the results, with their proper signs, to those already obtained.

[Note.—This rule merely directs us to perform mentally the operations actually placed before the eye in the illustration given above. There is certainly more work for the fingers in thus transcribing the several expressions under a different arrangement of its component terms, but the risk of mistake is undoubtedly less, as there is less hazard of overlooking *like* quantities. It may be as well to observe here that although it matters not in what order the terms of an algebraical expression are written, yet, as in addition, it is customary to collect the terms like those which head the several columns before those unlike them are collected, the directions in the rule are given accordingly.]

(1)

$$\begin{array}{l}
3x-5y+2z \\
-4z+3x-y \\
2y-z+a \\
3a+5z-2x \\
\hline
4x-4y+2z+4a \\
\hline
\end{array}$$

(2)

$$\begin{array}{l}
4ax-13y+3z \\
5y+3ax+9y \\
7xy-4z+90 \\
z+40-6y \\
\hline
7ax-5y+7xy+130 \\
\hline
\end{array}$$

Examples for Exercise.

(1)	(2)	(3)
$7ax-3by+14$	$13-2ax-mz$	$4abx-6cy+2abc$
$-2by+6+2ax$	$xy+3mz+4$	$84cy+abx-124$
$42-ax-by$	$2mz-ax-3$	$-17abc-3cy+2abx$
$-5ax-6by+18$	$-5ax+17+3mz$	$37-abc+4cx$

(4)	(5)	(6)
$18xy-12yz+3xyz$	$4\frac{x}{y}-3az+2b$	$\frac{7}{2}ax-\frac{3}{4}b+8$
$-4yz+5xyz-13yz$	$6b+2\frac{x}{y}-az$	$\frac{3}{4}ax+5b-4\frac{1}{2}$
$3xz-4xy+2yz$	$-2az-3b-\frac{x}{y}$	$6b-4ax-3c$
$7xyz+2xz-3xy$	$-5\frac{x}{y}+az-5b$	$4c+13-\frac{1}{2}b$

When the expressions to be added together are not already written one under another, as in these examples, but are placed side by side, as in examples 7, 8, 9, 10, below, then, instead of placing them as here, without any regard to like quantities being confined to one column, it is better, of course, in copying off the expressions for addition, to write the terms in such order that the operation may come under case 1, as is done in the examples at page 6. But, without arranging the terms in columns at all, you may, in general, easily select and add together the several sets of like quantities, by a careful examination of the row of expressions as they stand, though, to diminish the risk of mistake, the safest way is to bring all the expressions to be added under the arrangement of case 1. The following may be worked in either of these ways.

(7) Find the sum of $3ax-2b+1$, $-5b+3-2ax$, $-14+ax-9b$.

(8) $-9xy+5az-13$, $7az-17+4xy$, $10+2xy-3az$, $az+4$.

(9) $8xy-7az+3mn$, $-5az+3xy$, $7mn+2az-3xy$, $9xy-13+az$.

(10) $-3px+4qy-7a$, $5qy-a+2px$, $-2bx+3-7c$, $8a+4px+6qy$.

SUBTRACTION.

5. It has been seen that addition of algebra differs from the operation that goes by the same name in arithmetic, not only in the circumstance that in algebra letters stand for numbers, but also in this other, namely, that negative as well as positive numbers are introduced. The like difference has place between subtraction in algebra and subtraction in arithmetic. In arithmetic, all our operations are with

positive numbers or quantities ; in algebra, both positive and negative alike enter ; it is necessary, therefore, for the purposes of algebra, that we should know how to subtract a negative number.

Now, to *subtract* one thing from another, is to take away that thing, or its equivalent, from the other, and thus to get what is called the remainder ; thus, suppose we have to take 3 from 8, we may consider the 8 to be made up of 5 and 3, and therefore to be the same as $5 + 3$, and 3 being actually taken away, the remainder is 5. Suppose we have to take -3 from 8, we may, evidently, instead of 8, write $8 + 3 - 3$, and -3 being actually taken away, the remainder is $8 + 3$, or 11 ; we conclude, therefore, that if -3 be subtracted from 8, the remainder is 11. Again, suppose we have to take -3 from -8, then, writing for -8, its equivalent $-5-3$, and actually taking the -3 away, the remainder is -5 ; so that, if -3 be subtracted from -8, the remainder is -5. Lastly, suppose we have to subtract 3 from -8, writing for -8 its equivalent $-8 -3 + 3$, and then actually taking the 3 away, the remainder is $-8 -3$, or -11.

We have thus considered all possible varieties as to the signs of the two numbers, and have arrived at the following results.

Subtraction.

$$\begin{array}{rrrr} 8 & 8 & -8 & -8 \\ 3 & -3 & -3 & 3 \\ \hline 5 & 11 & -5 & -11 \\ \hline \end{array}$$

And it is easy to see that these same results would have been obtained if we had *changed the sign* of the number to be subtracted in each case, and had then *added* thus :—

Addition.

$$\begin{array}{rrrr} 8 & 8 & -8 & -8 \\ -3 & 3 & 3 & -3 \\ \hline 5 & 11 & -5 & -11 \\ \hline \end{array}$$

The two numbers 8 and 3 in the foregoing reasoning have been taken quite at random ; it is obvious that if any other two numbers had been chosen the conclusion would have been the same, namely, that *subtraction may always be converted into addition by simply changing the sign of the number to be subtracted ;* we thus deduce the following rule :—

RULE.—Change the sign of every quantity to be subtracted, or *conceive* it to be changed (without actually rubbing it out, and putting the opposite sign in its place), and then proceed as if the operation were addition instead of subtraction ; the result will be the correct remainder.*

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* Beginners often puzzle themselves a good deal about negative quantities, and, from regarding them in an erroneous light, surround them with a degree of perplexity from which they are naturally free. This perplexity is, perhaps, increased by describing them, as some

	(1)	(2)	(3)
From	$5ax+7by$	$6xy-3az$	$-2mx-3ny+4$
Take	$3ax-2by$	$-4xy+2az$	$5mx+2ny-3$
Remainder	$2ax+9by$	$10xy-5az$	$-7mx-5ny+7$

(4) Subtract $7x-3y+2a-6b-5$ from $5y-2x+3b-a-c$.

Here it will be advisable to alter the arrangement of the terms of the quantity to be subtracted (or of the other quantity, whichever you choose), so that the like quantities may stand the one under the other. It is also necessary to observe that those terms which have no like are to be annexed to the remainder, obtained from the other terms, by their proper signs; and, to be correct in these signs, you must, of course, change the sign of every term to be subtracted that is thus annexed: as follows—

$$\begin{array}{rr} & 5y-2x+3b-a\ \ -c \\ & -3y+7x-6b+2a-5 \\ \hline \text{Remainder} & 8y-9x+9b-3a-c+5 \\ \hline \end{array}$$

Examples for Exercise.

$$\begin{array}{c} (1) \\ 8xy-5az+b\ \ -1 \\ -3xy-2az+7b-4 \\ \hline \\ \hline \end{array} \qquad \begin{array}{c} (2) \\ -13xyz+4ab-3cd \\ 5xyz-8ab+2cd-4 \\ \hline \\ \hline \end{array}$$

---

teachers do, as "quantities less than nothing." A practical illustration, however homely, of an elementary difficulty in science, will often do more to make the matter clear than pages of abstract reasoning; I, therefore, offer the following, suggested by what is done in the text:—

Suppose a quantity of 8℔ weight is put into a scale-pan, it will, of course, balance 8℔ in the other scale-pan. Let now 3℔ be added to the quantity, but, at the same time, let an upward pressure of 3℔ act underneath the scale-pan; this opposing force or upward pressure would be correctly represented by -3℔, and the balance would, of course, remain undisturbed, that is, 8℔, acting on the scale-pan, is the same, in effect, as $8+3-3$℔ so acting. Now, if the -3℔ (that is, the upward pressure) be taken away, the result is $8+3$, or 11℔; hence, if -3 be taken from 8 (that is, from $8+3-3$) the remainder must be 11. And universally, whenever you take away a negative quantity, you diminish by so much the opposing influence, or increase by so much the direct influence—two forms of expression which may be allowed to amount, in meaning, to the same thing. In reference to the above illustration, there are actually upward and downward pressures acting on the scale-pan from the air. If we call the downward pressure a, the upward will be $-a$, and the two united will be $a-a$, or 0, so that the weight would be the same were the air removed.

(3)

$$\begin{array}{r} 6mx-3ny+4pz-5 \\ -5ny-4pz+10-a \\ \hline \\ \hline \end{array}$$

(4)

$$\begin{array}{r} ax-3by+c \\ -9z+ax-4by-2a+3m \\ \hline \\ \hline \end{array}$$

(5) Subtract $-8az-2by+5cx-3$ from $9az-by+3$

(6) $11px-10qy+4r$ from $-7px+13qy-11$

(7) $-3az+2by-6x+12$ from $18by+7x-2az-8$

(8) $7a-5b+6c-9x$ from $-8x-4c+b-7a+9$

(9) $-2xy+13az-2\frac{1}{2}$ from $-12az+\frac{xy}{2}$[*] $+3\frac{1}{3}$

(10) $3\frac{x}{a}-5\frac{x}{b}+2y$ from $2\frac{x}{a}-3y+\frac{x}{b}-4$

(11) $16-7ax+15by-\frac{3}{4}c$ from $5ax-7by+\frac{3}{8}c-11$

(12) $2\frac{1}{2}x-3\frac{1}{4}y+4z-\frac{7}{8}$ from $5\frac{1}{2}y-3\frac{1}{2}z+\frac{3}{4}$

(13) $-7\frac{1}{2}ay+\frac{x}{2}-9z+\frac{1}{4}$ from $2\frac{1}{2}x-4\frac{1}{2}ay+\frac{1}{2}$

(14) $3cex-\frac{1}{2}ay-\frac{7}{4}z$ from $3\frac{1}{4}ay-2\frac{1}{3}cex+\frac{3}{2}z-1$

(15) $-\frac{x}{2}+\frac{y}{3}-\frac{z}{4}+c$ from $3\frac{1}{2}x-2\frac{1}{3}y+5\frac{1}{4}z+a-b$

(16) $\frac{3x}{4}-\frac{4y}{5}+\frac{2z}{3}-8\frac{1}{3}$ from $\frac{z}{6}-\frac{x}{4}+\frac{y}{5}+48\frac{3}{4}$

On the use and management of brackets or vincula.

6. The signs + and −, in what has preceded, have been employed to connect together only simple quantities. They may be used also to connect together only compound quantities. In the examples just given certain compound quantities are subtracted from others; and this subtraction has been actually performed, as far, at least, as the operation was possible without a numerical interpretation of the letters. In example 4, for instance (page 10), $-3y$ has been actually subtracted from $5y$, and the remainder $8y$ exhibited; but the subtraction of the -5 from the $-c$, in the same example is not actually performed but

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[*] The learner will keep in mind that $\frac{xy}{2}$ is the same as $\frac{1}{2}xy$, that is, that the coefficient of xy in the expression in the text is $\frac{1}{2}$. In like manner, in the first expression in example 15, the coefficients of x, y, z are $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{4}$ respectively.

only indicated, because, without knowing the number that $-c$ represents, we cannot take the number -5 from it; we therefore merely write down $-c+5$, performing, in fact, no operation at all, except merely changing the subtraction into addition.

If we were to limit ourselves simply to *indicating* the subtraction in Ex. 4, without actually *performing* it, we should inclose the whole of the expression to be subtracted in *brackets*, and unite it, by the sign of subtraction, to the other quantity, thus :—

$$5y-2x+3b-a-c-(7x-3y+2a-6b-5)$$
$$\text{or } 5y-2x+3b-a-c-\{7x-3y+2a-6b-5\}$$
$$\text{or } 5y-2x+3b-a-c-[7x-3y+2a-6b-5]$$

The minus sign thus placed before a bracketed quantity denotes the subtraction of the whole of that quantity from what precedes; the brackets may always be removed provided the sign of every term included within them be changed, agreeably to the rule for subtraction, so that, upon removing the brackets, each of the above forms would be

$$5y-2x+3b-a-c-7x+3y-2a+6b+5$$

which expression, by actually performing the operations indicated by the signs, as far as these can be performed, we have seen to be

$$8y-9x+9b-3a-c+5$$

which might have readily been written down, as the equivalent of the foregoing row of terms, without the more formal process at p. 10.

If the bracketed expression were to be added to, instead of subtracted from, what precedes it, then the signs of its terms would remain unchanged, though the brackets be removed, because we do not interfere with the signs in *addition*. The following are some instances of the introduction and removal of brackets.

$$3ax-4by+(2by-ax)=3ax-4by+2by-ax=2ax-2by$$
$$3ax-4by-(2by-ax)=3ax-4by-2by+ax=4ax-6by$$
$$3ax-4by+(-2by-ax)=3ax-4by-2by-ax=2ax-6by$$
$$3ax-4by-(-2by-ax)=3ax-4by+2by+ax=4ax-2by$$
$$\begin{aligned}2ax+\{ax-(2y-3ax)\}=2ax+\{ax-2y+3ax\}&=2ax+4ax-2y\\&=6ax-2y\end{aligned}$$
$$\begin{aligned}2ax-\{ax-(2y-3ax)\}=2ax-\{ax-2y+3ax\}&=2ax-4ax+2y\\&=2y-2ax\end{aligned}$$

Besides the brackets ( ), [ ], { }, a straight line or bar drawn over the quantities to be linked together is sometimes, though much less frequently, employed; thus, $x+y+\overline{x-y}$ and $x+y-\overline{x-y}$ are respectively the same as $x+y+(x-y)$ and $x+y-(x-y)$. It may be as well to notice here that the two expressions thus chosen to show the meaning of the bar-vinculum, furnish a general property of numbers which is often in request: it is this, namely—

$$x+y+(x-y)=2x, \text{ and } x+y-(x-y)=2y$$

two statements which are, of course, true, whatever numbers x and y

stand for, so that the sum of two numbers increased by their difference gives twice the greater of the two; and the sum diminished by the difference gives twice the less. Suppose, for instance, x were $=13$, and $y=7$, then the sum 20, increased by the difference 6, is $26=13\times 2$; and the sum 20, diminished by the difference 6, is $14=7\times 2$. Whatever bond or tie be employed to unite a set of quantities into one compound whole, whether the bracket or the bar, the tie so employed is called a *vinculum*, without reference to its peculiar form. A multiplier or coefficient placed before a set of quantities thus united together by a vinculum, implies that each individual term in the compound whole is to be multiplied by that coefficient, so that it must be introduced as a factor in every simple term when the vinculum is removed, thus:—

$$a+6\,(b-c+d)=a+6b-6c+6d.$$

$$3\,(x-4)+5\,(y+2)=3x-12+5y+10=3x+5y-2,\ \text{\&c.}$$

And conversely when the same factor is found to enter two or more of the simple terms of an expression, the repetition of this common factor is avoided by introducing a vinculum, thus, taking the results at p. 12, they may be written as below, and they are more convenient for actual computation when numbers are put for the letters, in this changed form:—

$$\begin{array}{ll} 2ax-2by=2\,(ax-by) & 6ax-2y\ \ =2\,(3ax-y) \\ 4ax-6by=2\,(2ax-3by) & 2y\ -2ax\ =2\,(y-ax) \\ 2ax-6by=2\,(ax-3by) & 3ax+9abx=3ax\,(1+3b) \end{array}$$

Suppose, in this last example, a, b, and x were 5, 6, and 7 respectively; you will see in a moment that the expression on the right of the sign of equality is much more readily calculated than that on the left, since the work of the latter is $3\times 5\times 7+9\times 5\times 6\times 7$, whereas that of the former is only $3\times 5\times 7\ (1+3\times 6)$ that is, $3\times 5\times 7\times 19$, and if a, b, x had represented large numbers, the saving of work, by taking the form on the right instead of that on the left, would be very considerable. And this is one object had in view in introducing vincula, or brackets—the saving of numerical work.

Write the following expressions free from brackets—

(1) $3a-2\,(a-b)$ (2) $4\{3a-(b-a)\}$ (3) $2\{2x-(x+y)-1\}$

(4) $5[a+x-2(x-a)]$ (5) $2\{a-3\,(2x-3)\}$ (6) $3\{2a-(b-3a+c)\}$

The equivalents of these expressions are (1) $a+b$, (2) $16a-4b$, (3) $2x-2y-2$, (4) $15a-5x$, (5) $2a-12x+18$, (6) $15a-3b-3c$, and these, after (1), may be put in the following more convenient forms; (2) $4\,(4a-b)$, (3) $2\,(x-y-1)$, (4) $5\,(3a-x)$, (5) $2\,(a-6x+9)$, (6) $3\,(5a-b-c)$.

SIMPLE EQUATIONS.

7. An *equation* is merely a statement in algebraical characters, or in the figures of arithmetic, that two quantities are *equal*; thus, if x be such a number that $x+2$ is equal to 7, then the statement $x+2=7$ is an equation; it is plain that for this to be true x must represent 5;

in like manner, $2x-3=9$ is an equation, it affirms that x denotes a number such that the double of it diminished by 3 is 9; it is pretty obvious that, in order that such may be the case, x must stand for 6. An equation is said to be *solved* when the value of the letter, at the outset unknown, is discovered; thus, the solution of the equation $x+2=7$ is $x=5$; and the solution of the equation $2x-3=9$ is $x=6$. The letters chosen by algebraists to represent numbers that are at first unknown are taken from towards the end of the alphabet, as z, y, x, &c., and when for brevity they put letters for known numbers, they choose them from the beginning of the alphabet, as a, b, c, &c.

In the equations which follow, the unknown quantity—that is, the number to be determined—will usually be represented by x; so that to solve the equation will be to find the value of x. The two operations in principal request for this purpose are called *Transposition* and *Clearing Fractions*.

Transposition consists in simply taking a quantity away from one side of an equation and placing it on the other side, at the same time taking care that the perfect equality of the two sides be undisturbed, so that the result may still be an equation. In order to this, the quantity taken from one side must have *its sign changed* when it is transposed to the other side; if this change of sign be neglected, the balance of the two sides will be disturbed; if it be attended to, the balance will be preserved, as will be sufficiently evident from the following illustrations.

Suppose we have the equation $6x-4=5x+2$, in which the $6x$ and the $5x$ are unknown terms, and the 4 and the 2 known. In order to solve this equation, that is, to find the value of x, it is necessary first that *all* the unknown terms should be collected together on one side of the equation, and all the known on the other; we have, therefore, to remove the -4 from the left-hand side and the $5x$ from the right, and to do this by such operations as will not disturb the equality of the two sides. Now, let 4 be added to each side, or, which is the same thing, let -4 be subtracted from each; the result will, of course, be an equation, namely, the equation $6x=5x+2+4$, because if equals be added to equals the results must be equal. The new equation, thus deduced, is nothing but the original with the -4 on the left taken away and written, with changed sign, on the right. Subtract now $5x$ from each side of the equation just deduced, the result must still be an equation, namely, the equation $6x-5x=2+4$, because if equals be taken from equals the results must be equal. Now, by comparing this last equation with that originally proposed, you will see that what has been done to obtain it is, in fact, merely the transposing the -4 from the left-hand side to the right, and the $5x$ from the right to the left, the *signs* of the quantities thus transposed being changed.

As the equation $6x-5x=2+4$ is the same as $x=6$, the value of x is in this way discovered to be 6. The original equation affirmed that 6 times some number (represented at the outset by x), diminished by 4, is equal to 5 times the same number increased by 2; the number is found, as above, to be 6.

Again; let the equation proposed for solution be $7x-3=4x+9$. By adding 3 to each side, or which is the same thing, subtracting -3

from each side, the result is $7x = 4x + 9 + 3$. By subtracting $4x$ from each side of this, we have $7x - 4x = 9 + 3$; and comparing this with the original you see that the -3 and the $4x$ have been transposed, and their signs changed. And it is obvious that the removal of any term from one side of an equation, by subtracting the equal of that term from *both* sides, is effected by, and is the same thing as, transposing that term with changed sign to the other side of the equation, the subtraction of a quantity being the same as its introduction with changed sign. The equation $7x - 4x = 9 + 3$, just deduced, is the same as $3x = 12$; if we divide each of these equals by the coefficient 3, there results $x = 4$; and thus the value of x, at first unknown, is determined; we learn that the number, such that 7 times that number diminished by 3 may be equal to 4 times the same increased by 9, must be 4.

Whenever a fraction occurs in an equation, we may deduce another equation from which the fraction shall be absent by simply multiplying both sides of the equation by the denominator of that fraction. When every fraction is removed the equation is said to be cleared of fractions.

Suppose the equation to be $\frac{x}{3} + 2x = 7$; then multiplying each side by 3, there results the equation $x + 6x = 21$; for equal quantities, whether multiplied or divided by the same number, furnish equal results. The equation just deduced is $7x = 21$, therefore, dividing each side by 7, we get $x = 3$, which value put for x satisfies the original equation, for $\frac{3}{3} + 2 \times 3 = 7$.

To solve a simple equation containing only one unknown quantity.

8. Rule 1. If a fraction occurs in the equation, clear it away by multiplying both sides by its denominator.

2. If both known and unknown quantities occur on the same side, separate them by transposition, so as that another equation may be deduced in which all the unknown quantities are on one side and all the known on the other.

3. Collect the quantities on each side into a single term, namely, a single unknown term on one side and a single known term on the other.

4. If the coefficient in this unknown term (that is, the coefficient of x) be other than unit, divide each side by that coefficient; then x alone will stand on one side of the equation, and the known number, which is its interpretation, on the other.

Example 1. Given the equation $7x + 3 = 2x + 13$, to find the value of x.

As no fractions enter this equation, the first precept of the rule is not required; but as known and unknown quantities are connected together on each side, we must apply the second precept of the rule and separate them. By transposition, $7x - 2x = 13 - 3$
that is, collecting the terms, $5x = 10$.

As the coefficient of x in this last equation is 5, we divide each side by 5, agreeably to precept 4, and we thus get finally $x = 2$; the value required.

(2) Given the equation $8-3x+12=30-5x+4$ to find the val
By transposition, $5x-3x=30+4-8-12$
Collecting the terms, $2x=14$
Dividing by 2, $x=7$

(3) Given $8-\frac{5x}{3}+4x=22$, to find the value of x.
Multiplying by 3, $24-5x+12x=66$
Transposing, $-5x+12x=66-24$
Collecting, $7x=42$
Dividing by 7, $x=6$

(4) Given $\frac{x}{4}+\frac{2x}{3}-1=4-x$ to find the value of x.
Multiplying by 4, $x+\frac{8x}{3}-4=16-4x$
Multiplying by 3, $3x+8x-12=48-12x$
Transposing, $3x+8x+12x=48+12$
Collecting, $23x=60$
Dividing by 23, $x=\frac{60}{23}=2\frac{14}{23}$

(5) Given $\frac{2x}{3}+\frac{3x}{4}-\frac{x}{5}=73$ to find the value of x.
Multiplying by 3, $2x+\frac{9x}{4}-\frac{3x}{5}=219$
Multiplying by 4, $8x+9x-\frac{12x}{5}=876$
Multiplying by 5, $40x+45x-12x=4380$
Collecting, $73x=4380$
Dividing by 73, $x=\frac{4380}{73}=60$

(6) Given $\frac{x}{2}+\frac{x}{6}-\frac{x}{4}-\frac{1}{2}=0$, to find the value of x.
Multiplying by 2, $x+\frac{x^{*}}{3}-\frac{x}{2}-1=0$
Multiplying by 3, $3x+x-\frac{3x}{2}-3=0$
Multiplying by 2, $6x+2x-3x-6=0$
Transposing, $6x+2x-3x=6$
Collecting, $5x=6$
Dividing by 5, $x=\frac{6}{5}=1\frac{1}{5}$

[In the foregoing specimens of the method of solving a simpl tion with one unknown quantity, the precepts of the rule ha strictly obeyed; but after a little practice in the operation, you able to put some of the steps in a more compact form. Brevity has not been regarded in what is done above, only clearness and simplicity.]

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* The learner will bear in remembrance that a fraction is mu by a number when its denominator is divided by that number.

Examples for Exercise.

Find the value of x in each of the following equations.

(1) $5x-8=3x+2$

(2) $7x+3=42-6x$

(3) $2x+\frac{3x}{4}-66=0$

(4) $\frac{x}{3}-\frac{x}{4}=13-x$

(5) $3x+\frac{x}{2}+2=5x-4$

(6) $\frac{x}{3}+\frac{x}{4}+\frac{x}{5}-1=0$

(7) $2(x-3)=8$

(8) $3(x+1)=5(x-1)$

(9) $\frac{x-2}{3}=\frac{5x-11}{4}$

(10) $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$*

(11) $\frac{3(x-2)}{2}+6=3(x-3)$

(12) $2x-\frac{85-15x}{2}+14=0$

(13) $\frac{1}{3}x+\frac{1}{5}x=\frac{1}{4}x-\frac{1}{2}$

(14) $\frac{1}{2}(x-3)+5=\frac{2}{3}(x+4)-8$

(15) $\frac{x}{5}-\frac{2(3-2x)}{7}=12$

(16) $\frac{2x+3}{5}-\frac{7-3x}{2}=\frac{1}{2}(x+9)$

(17) $5\left(\frac{2x}{3}-\frac{x}{6}+1\right)=4x-3$

(18) $3x-2+4(-2x)=10-(x-1)$

(19) $2x-5(12-x)+3(2x-3)=0$

(20) $8\{2x-(16-5x)+4\}=5(3x-2)$

Application of Simple Equations to the Solution of Questions.

9. A question is solved by algebra by first expressing the conditions of the question in the form of an equation, and then solving the equation, as in the foregoing examples. The unknown quantity, or that which is to be discovered as the answer to the question, is represented, at the commencement, by an algebraic symbol, as x; and when the question is translated into an equation, all we have to do is to find the value of x; the mode of proceeding will become sufficiently clear by carefully examining the steps in the solutions here following.

NOTE. The symbol $\therefore$ stands for the word *therefore;* it is very frequently used, for brevity, in algebraical inquiries, and the same symbol reversed, thus $\because$, is sometimes employed for the word *because*.

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* The line which separates the two terms of a fraction, that is, the numerator and denominator, answers the purpose of a vinculum whenever either of those terms is a compound quantity, so that the sign placed before the fraction must be understood as extending its influence to the *whole* of the numerator; when the denominator 3 is cleared or removed from the fraction above, then, since the minus sign affects the whole of the numerator, the result must be written $-(x-10)$, or, $-x+10$, and similarly in Ex. 12. The learner must never forget that the minus sign before a vinculated quantity is a direction to change all the signs in that quantity upon the removal of the vinculum.

1. Find two numbers such that their sum may be 15, and their difference 7.

Let x stand for one of the numbers, then $15-x$ must represent the other, and the difference of the two will be expressed by $15-x-x$; this difference, we are told, is 7, hence the equation to be solved is

$$15-x-x=7$$

that is, $$15-2x=7$$

Transposing, $$15-7=2x$$

that is, $$8=2x, \therefore 4=x$$

consequently, one of the required numbers is 4, and since the other is $15-x$, that is $15-4$, or 11, it follows that the two numbers are 4 and 11.

In this solution, it is assumed that x represents the smaller of the two numbers; if it be made to represent the greater, then the other number $15-x$ must be subtracted from it, and the equation to be solved will be

$$x-(15-x)=7$$

Removing the vinculum, $$x-15+x=7$$

Transposing, $$2x=7+15=22$$

$$\therefore \quad x=11$$

consequently, $15-x=15-11=4$, $\therefore$ the two numbers are 4 and 11.

2. A railway-train leaves London at 12 o'clock to run to York, a distance of 200 miles, another leaves York at the same time for London: at what o'clock will the trains pass each other, supposing that the train from London travels at the rate of 25 miles an hour, and that from York at the rate of 35 miles an hour?

Suppose they meet in x hours, then the train from London will have travelled $25x$ miles, and that from York $35x$ miles; and, as the sum of these distances make up the whole 200 miles, we, therefore, have the equation

$$25x+35x=200$$

that is, $$60x=200$$

$$\therefore x=\frac{200}{60}=\frac{20}{6}=3\tfrac{1}{3}$$

consequently, the trains will pass each other at $3\frac{1}{3}$ hours after starting, that is, at 3 h. 20 min. past 12 o'clock.

3. Divide the number 20 into two such parts, that 3 times one part added to 5 times the other part, may make 76.

Let x be one part, then, by the question, $20-x$ must be the other part, and, from the other conditions of the question we have the equation

$$3x+5(20-x)=76$$

Removing the vinculum, $$3x+100-5x=76$$

Transposing. $$3x-5x=76-100$$

$$\therefore \quad -2x=-24$$

Dividing by -2, $$x=12$$

consequently, since $20-x=20-12=8$, the two parts are 12 and 8.

You see that the sum of these makes 20, and that 3 times the first added to 5 times the second makes 76.

4. What number is that of which the third part exceeds the fourth part by 26?

Let x be the number; its third part is $\frac{x}{3}$, and its fourth part is $\frac{x}{4}$; and by the question

$$\frac{x}{3}-\frac{x}{4}=26$$

Multiplying by 3, to clear the first fraction, we have the equation

$$x-\frac{3x}{4}=78$$

Multiplying by 4, to clear the second fraction, we have the equation

$$4x-3x=312$$

that is, $$x=312$$

consequently the number sought is 312; the third part of it is 104, and the fourth part 78; and the former part exceeds the latter by 26.

5. Divide £100 among three persons, A, B, C, so that A may have £20 more than B, and B £10 more than C.

Let x represent the number of pounds A should have; then by the question, the number in B's share would be $x-20$, and in C's, $x-20-10$; and since the sum of these numbers is to make 100, we have, by addition,

x	number of	pounds in	A's	share.
$x-20$	„	„	B's	„
$x-20-10$	„	„	C's	„
$3x-40-10=100$		„	all the shares	

that is, $3x-50=100$

Transposing, $3x=100+50=150$

$$\therefore\ x=50$$

consequently A must have £50; therefore B must have £30, and C £20.

6. A person pays a bill of £2. 16*s*. 6*d*. with forty pieces of silver, half-crowns and shillings; how many pieces were there of each sort?

Let x represent the number of half-crowns; then, by the question, $40-x$ was the number of shillings; so that the number of sixpences in the former was $5x$, and the number in the latter $2(40-x)$ or $80-2x$; also, the number of sixpences in the whole sum paid, namely, £2. 16*s*. 6*d*., is 113.

$$\therefore\ 5x+80-2x=113$$

Transposing, $5x-2x=113-80$

that is, $3x=33$

$$\therefore\ x=11$$

consequently there were 11 half-crowns, and therefore $40-11$ or 29 shillings.

7. A vessel can be filled from a tap in 3 hours, and from a second tap in 4 hours; in what time will it be filled if both taps be set running together?

Suppose it can be filled in x hours; then, since $\frac{1}{3}$ of it is filled in 1 hour by the first tap and $\frac{1}{4}$ of it by the second, $\frac{1}{3}+\frac{1}{4}$, that is, $\frac{7}{12}$ of it is filled by both in 1 hour, consequently x times this, or $\frac{7}{12}x$ is filled in x hours; but by the supposition this is the whole vessel, that is, 1 whole vessel.

$$\therefore \quad \frac{7x}{12}=1 \quad \therefore \quad 7x=12 \quad \therefore \quad x=\frac{12}{7}=1\tfrac{5}{7}$$

consequently the vessel will be filled in $1\frac{5}{7}$ hours.

8. A vessel can be filled by taps running separately in 2 hours, 3 hours, 4 hours, and 5 hours; in what time will it be filled if they all run together?

The part of the whole vessel filled by all the taps running one hour is $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ or $\frac{30}{60}+\frac{20}{60}+\frac{15}{60}+\frac{12}{60}=\frac{77}{60}$, consequently the part filled by them in x hours is $\frac{77x}{60}$; if this part be in reality the whole, or 1 vessel, then we have the equation $\frac{77x}{60}=1$

$$\therefore \quad 77x=60 \quad \therefore \quad x=\frac{60}{77}$$

so that the vessel will be filled in $\frac{60}{77}$th part of an hour.

In this way, when we know the time in which any number of agents acting singly can produce an effect, we may always find the time in which the same effect would be produced by all the agents acting collectively.

9. A market woman bought a certain number of apples at the rate of five for two-pence; she sold one-half of them at two a penny, and the other half at three a penny, and thus gained 3*d.*; how many apples did she buy?

Suppose she bought x apples; then, as she gave 2*d.* for every 5, they must have cost $\frac{2x}{5}$ pence.

Again, for $\frac{x}{2}$ at two a penny she received $\frac{x}{4}$ pence, and for $\frac{x}{2}$ at three a penny she received $\frac{x}{6}$ pence. Consequently, as she received 3*d.* more than she spent,

$$\frac{x}{4}+\frac{x}{6}=\frac{2x}{5}+3$$

Multiplying by 4, $x+\frac{2x}{3}=\frac{8x}{5}+12$

Multiplying by 3, $3x+2x=\frac{24x}{5}+36$

that is, $5x=\frac{24x}{5}+36$

Multiplying by 5, $25x=24x+180$

Transposing, $x=180$

consequently, the number bought was 180, the cost of these at five for two-pence was $\frac{180}{5} \times 2 = 72$ pence, the selling price of 90 at two a penny was $\frac{90}{2} = 45$ pence, and the selling price of 90 at three a penny was $\frac{90}{3} = 30$ pence, so that the selling price of the whole 180 was 75 pence, three-pence more than the cost price.

NOTE. When, as in the preceding question, fractions are foreseen to enter the operation when x is put for the unknown number, it will often be found to simplify the work to put instead of x some convenient multiple of x, that shall actually be divisible by the denominators; thus, as 5 and 4 and 6 are easily foreseen from the above question to be divisors of the unknown number, let that number be represented by $60x$, instead of by x; then the cost price of the $60x$ apples is $\frac{60x}{5} \times 2 = 24x$ pence, the selling price of $30x$ at 2 a penny is $15x$ pence, and of $30x$ at 3 a penny, $10x$ pence; hence the equation is

$$15x + 10x = 24x + 3$$

Transposing, $25x - 24x = 3$

that is, $x = 3 \therefore 60x = 180$

Example 4 might have been simplified in the same manner, thus—

Let $12x$ be the number, then the third part of it is $4x$, and the fourth part $3x$, therefore by the question

$$4x - 3x = 26$$

that is, $x = 26 \therefore 12x = 312$, the number required.

10. Out of a cask of wine which had leaked away one-fourth part, 30 gallons were drawn, it was then found to be one-half full; how much did it hold?

Suppose it held $4x$ gallons, then after the leakage it contained $3x$ gallons; hence by the question

$$3x - 30 = 2x$$

Transposing, $3x - 2x = 30$

that is, $x = 30 \therefore 4x = 120$, the number of gallons.

Questions for Exercise.

1. Find two numbers such that their sum shall be 10, and their difference 6.

2. What number is that whose fourth part exceeds its fifth part by 10?

3. Divide 20 into two such parts, that a third of one part added to a fifth of the other may make 6.

4. Divide £300 among A, B, C, so that A may receive twice as much as B, and C as much as A and B together.

5. How much tea at 4*s.* 6*d.* per ℔, must be mixed with 50℔ at 6*s.* per ℔, so that the whole may be worth 5*s.* per ℔?

6. A labourer was engaged for 30 days, on condition that for every day he worked he was to be paid 20*d.*, but for every day he idled 10d. was to be forfeited. He received £1: how many days did he work?

7. A fish was caught whose tail weighed 9lb; his head weighed as much as his tail and half his body, and his body weighed as much as his head and tail together: required the weight of the fish.

8. Out of a cask of wine, of which a fifth part had leaked away, 10 gallons were drawn, after which it was found to be two-thirds full: how much did the cask hold?

9. A person has £50 in half-guineas and crowns; the whole number of coins is 101: how many are there of each?

10. A gamester at the first sitting lost one-fifth of his money and then won 10 shillings; at a second sitting he lost one-third of what he then had, and won 3 shillings; he then found himself in possession of £3 3*s.*: how much had he at first?

11. A cistern can be filled from two pipes; from the first in 3 hours, and from the second in 5 hours: in what time can it be filled, if both pipes run together?

12. A can finish a piece of work in 2 days, B in three days, and C in 5 days: in what time will it be finished, if all three work together?

13. A child being asked the time by the clock, said, that the hands were both together between 5 and 6: required the exact time.

14. What is the time between 12 and 1 o'clock when the hour-hand points directly opposite to the minute-hand?

15. A person wishes to make a present of six guineas to a family of four persons; to the eldest he intends to give sovereigns, to the next half-sovereigns, to the third shillings, and the youngest sixpences, and to each the same number of coins: it is required to find how many each should have.

16. A and B working together can reap a field in 7 days, which A alone can reap in 10 days: in how many days can B alone reap the field?

17. A walks at the rate of 3 miles an hour; two hours after he has left, B is despatched, at the rate of 4 miles an hour, to overtake him: at what distance from the starting-point will B come up with A?

18. Two persons, 150 miles apart, set out at the same time to meet each other; one goes three miles while the other goes 7 miles: what part of the distance will each have travelled when they meet?

19. Find two numbers whose difference is 6, such that one-third the less increased by one-fifth the greater shall be the same as one-third the greater diminished by one-fifth the less.

20. A and B can finish a piece of work in 3 days, A and C in 5 days, and B and C in 6 days: in what time can they finish it, if A, B, and C all work at it together?

10. From the foregoing exercises it will be seen that a very small amount of algebraical knowledge will enable you to solve questions that would be found very puzzling if they were to be worked by arithmetic only. As yet addition and subtraction of algebra are the only rules that have been taught; multiplication and division are now to be considered, but some additional definitions must first be given.

DEFINITIONS.

11. *Powers.* When equal factors are multiplied together, the product is called a *power* of the factor thus repeated. The second power is *produced from two* equal factors, the third from three, the fourth from

four, and so on. Thus, because $3 \times 3 = 9$, 9 is the second power or *square* of 3; and because $3 \times 3 \times 3 = 27$, 27 is the third power or *cube* of 3. In like manner, 27×3, or 81, is the fourth power of 3, and so on. In the same way with letters, aa is the square of a, aaa is the cube of a, $aaaa$ is the fourth power of a, &c.

Exponents, or *Indices*. Instead of writing down all the equal factors producing a power side by side as above, it is sufficient to write only one factor, and to mark the number of them by a small figure placed over the right-hand corner of it; thus—

$$3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243,\ 3^6 = 729,\ \&c.$$

The left-hand members of these equations denote the second (or square), the third (or cube), the fourth, fifth, sixth, &c., powers of 3; and the right-hand members show what these powers really are. So with respect to letters; a^2, a^3, a^4, a^5, a^6, &c., denote the same powers of a, that is, of the number which a stands for. The small corner figures are called *exponents*, or *indices*. In the case of numbers, we can of course actually exhibit the power by multiplying together as many of the equal factors as the exponent points out; in the case of letters we can do no more than indicate the power, as above, till the letter is replaced by the number it stands for, when the operation indicated may be performed by common multiplication.

Roots. The number to which the exponent is attached is the corresponding root of the resulting power; thus, 3 is the square root of 9, the cube root of 27, the fourth root of 81, &c. When the root is known, it is easy enough to get the power by common multiplication; but when the power is given to determine the corresponding root, the operation is frequently one of difficulty; you have seen in arithmetic, that to find the square root, the cube root, &c. of a number is a much more troublesome operation than to find the square or cube of a number. Roots, like powers, are indicated by exponents; thus, $9^{\frac{1}{2}}$, $27^{\frac{1}{3}}$, $81^{\frac{1}{4}}$, $a^{\frac{1}{5}}$, &c., denote respectively the square root of 9, the cube root of 27, the fourth root of 81, the fifth root of a, &c. Roots are otherwise denoted by means of what is called *the radical sign* $\surd$; thus, the roots just expressed may be indicated as follows:—

$$\sqrt{9},\ \sqrt[3]{27},\ \sqrt[4]{81},\ \sqrt[5]{a},\ \&c.$$

The mark $\surd$ is merely a corruption of the letter *r*, the initial of the word *root* (or *radix*); when the square root is denoted, the little figure 2, implying the *second* root, is suppressed, as its insertion would be superfluous; the absence of an index from the radical sign sufficiently indicates that the square root is meant.

The signs of operation here defined, when connected with compound or bracketed quantities, just like the other signs of operation attached to such quantities, operate upon the whole expression; the bracket or vinculum may always be regarded as reuniting into one term what had previously been cut up into several terms. The square of the compound quantity $ax^2 - bx + c$ is indicated by help of the vinculum in either of the four ways following—

$$(ax^2 - bx + c)^2,\ [ax^2 - bx + c]^2,\ \{ax^2 - bx + c\}^2,\ \overline{ax^2 - bx + c}^{\,2}$$

and the square root of the same expression is denoted by one or other of the forms

$$\sqrt{}(ax^2-bx+c),\ \sqrt{}[ax^2-bx+c],\ \sqrt{}\{ax^2-bx+c\},\ \sqrt{ax^2-bx+c}$$

or else by

$$(ax^2-bx+c)^{\frac{1}{2}},\ [ax^2-bx+c]^{\frac{1}{2}},\ \{ax^2-bx+c\}^{\frac{1}{2}},\ \overline{ax^2-bx+c}^{\frac{1}{2}}$$

Add together the following expressions—

1. $(a+x)^2-3,\ \{a+x\}^2+6,\ [a+x]^2-1,\ 5+4(a+x)^2$
2. $6-\sqrt{}(a-x),\ 3\sqrt{a-x}+2,\ 5(a-x)^{\frac{1}{2}}-4,\ 2-3(a-x)^{\frac{1}{2}}$
3. Subtract $4(2ax+3)^{\frac{1}{3}}-3b$ from $7\sqrt[3]{}(2ax+3)+2b$
4. Subtract $3\sqrt{x^2-y^2}-2(a+x)$ from $4(x^2-y^2)^{\frac{1}{2}}+7(a-x)$
5. Subtract $3a\sqrt{}(x^2+y^2)-2\sqrt{x^2-y^2}$ from $4(x^2-y^2)^{\frac{1}{2}}+2a(x^2+y^2)^{\frac{1}{2}}$

MULTIPLICATION.

Case 1. *When the factors are simple quantities.*

12. Rule 1. Notice the *signs* of the two factors:—if they are *like*, that is, both *plus* or both *minus*, the sign of the product is *plus;* if they are *unlike*, the sign of the product is *minus*. This is briefly expressed by saying—like signs produce plus, and unlike signs produce minus in the product.

2. Having written down the proper sign, write after it the product of the coefficients.

3. And then annex the product of the letters; that is, write down the letters in both factors side by side without any sign between them.

Note.—The letters of the product are usually written in the order they follow one another in the alphabet.

When powers of the same quantity are to be multiplied together, the product is expressed at once by simply writing down that quantity with the *sum* of the exponents in the factors over it; thus, $x^3\times x^5=x^8$, $x^2\times x^7=x^9$, and generally $x^m\times x^n=x^{m+n}$, whatever integers m and n may be; this follows from the very notation for powers explained at page 23; for $x^3\times x^5$ is $xxx\times xxxxx=x^8$, and similarly whatever be the exponents of x in the factors.

That the product of two quantities with like signs is plus, and the product of two with unlike signs minus, may be proved thus:—

Let us take any two numbers for factors, as, for instance, 6 and 4, and let us give to them all the possible varieties as respects signs; we shall then have to consider these four cases, namely—

$$\begin{array}{rrrr} 6 & -6 & 6 & -6 \\ 4 & 4 & -4 & -4 \\ \hline 24 & -24 & -24 & 24 \\ \hline \end{array}$$

The first is the case of common arithmetic, and requires no remark here. The product in the second case must be -24, because from the nature of multiplication the -6 is to be repeated 4 times. The third case is peculiar; in order to discover the correct product let the multiplier be increased by 5, the product will then, evidently, be 5 times 6, or 30 too great; if, therefore, we can find this erroneous product, we may correct it, and arrive at the true product by subtracting 30. Now -4 increased by 5, that is, $-4+5$, is 1, and $6\times1=6$, subtracting 30 from this erroneous product, we have $6-30=-24$, $\therefore -24$ is the correct product in the third case.

In like manner, in the fourth case, increase the multiplier by 5, thus getting a product which is 5 times -6, that is, -30 too great; this product is $-6\times1=-6$, and subtracting the -30, the correct product is $+24$, because we must change the sign of the subtractive quantity, and *add*. The products exhibited above are, therefore, all correct, and as any other factors might have been chosen, and similar conclusions obtained, it follows that the "rule of signs" is correct.

	(1)	(2)	(3)	(4)
Multiply	$3ax$	$-7cy$	$5x^2y$	$-8ax^2y^3$
by	$2b$	$3ax$	$-6ay^3$	$-4x^3y^2$
Product	$6abx$	$-21acxy$	$-30ax^2y^4$	$32ax^5y^5$

	(5)	(6)	(7)	(8)
Multiply	$-13a^2bx^{\frac{1}{2}}$	$3x\sqrt{y}$	$7a^2b^3z^4$	$abx^{\frac{1}{4}}$
by	$-5by$	$-5ax^5$	$5x^2z$	$-2by^{\frac{1}{2}}$
Product	$65a^2b^2x^{\frac{1}{2}}y$	$-15ax^6\sqrt{y}$	$35a^2b^3x^2z^5$	$-2ab^2x^{\frac{1}{4}}y^{\frac{1}{2}}$

Examples for Exercise.

(1) $4a^2x^3\times-3b^3xy^4$ (2) $-6ab^4xy^3\times-5x^4y^2z^3$

(3) $-2ax\times3a^2x^3\times5x^2y$ (4) $5abx\times-2b^2x^4\times-3y^2\times-4$

(5) $7x^2y^2\times\frac{1}{2}axy\times-\frac{3}{7}a^2$ (6) $-\frac{1}{2}ay^2\times-\frac{4}{3}bx^2\times-\frac{1}{4}az^2$

CASE II. *When one of the factors is a compound quantity and the other a simple quantity.*

13. **RULE.** Multiply each term in the compound factor by the simple factor, always commencing the operation at the left hand; the products, written one after the other, with their proper signs, will be the complete product.

	(1)	(2)
Multiply	$3ax-12by$	$4ax^2-3by^3+2z^4$
by	$2a^2x^2$	$-3bz$
Product	$6a^3x^3-24a^2byx^2$	$-12abx^2z+9b^2y^3z-6bz^5$

Examples for Exercise.

	(1)	(2)
Multiply	$-2xy^2+3az^2$	$12a^3b^2x-3cy^3+2$
by	$5ax^2$	$-7ab^3x^2$
Product		

(3) $(6ab^2x^2-2x^3y^2)\times 2a^2b^2x^2$

(4) $(7x^2y^2+az^2-4)\times -3bxy^3$

(5) $(9x^3-10y^4-12z^5)\times -7x^3y^4z^5$

(6) $\{ax-(3ax+2)\}$* $\times -4a^2bx^3$

(7) $4aby\{3a^2x-(4-4a^2x)\}$

(8) $\frac{1}{2}a^2x^2\{2by^2-(4ax-3y^2)\}$

(9) $(5ax+by-cz)\times -ax\times -bc$

(10) $(7ax^{\frac{1}{2}}-by)\times -ay^2\times bz^3$

CASE III. *When both the factors are compound quantities.*

14. RULE 1. Multiply *all* the terms of the multiplicand by *each* term of the multiplier.

2. Place the several *like* products one under another, and find their sum by addition.

(1) Multiply $a+b$ by $a+b$, and $a-b$ by $a-b$, as also $a+b$ by $a-b$.

$$\begin{array}{rl}
 & a+b \\
 & a+b \\
\hline
 & a^2+ab \\
 & \qquad ab+b^2 \\
\hline
(a+b)^2= & a^2+2ab+b^2 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{ll}
a-b & \\
a-b & \\
\hline
a^2-ab & \\
\qquad -ab+b^2 & \\
\hline
a^2-2ab+b^2 & =(a-b)^2 \\
\hline
\end{array}$$

$$\begin{array}{rl}
 & a+b \\
 & a-b \\
\hline
 & a^2+ab \\
 & \qquad -ab-b^2 \\
\hline
(a+b)\,(a-b)\ = & a^2 \qquad -b^2 \\
\hline
\end{array}$$

From these three results, we learn that

1. The square of the sum of two numbers is equal to the squares of the numbers themselves *plus* twice their product.

2. The square of the difference of two numbers is equal to the squares of the numbers themselves *minus* twice their product.

---

* In examples like (6), (7), and (8), the compound factor is to be prepared for multiplication by first removing the inner vinculum; thus, the first factor in Ex. (6) is to be written $ax-3ax-2$, that is, $-2ax-2$, and then multiplied by $-4a^2bx^3$. In like manner, the second factor in (7) is $7a^2x-4$.

3. The sum of two numbers multiplied by their difference is equal to the difference of their squares. Thus, take the two numbers 7 and 3; their sum is 10, and $10^2=7^2+3^2+21\times 2=100$.

The difference of the same numbers is 4, and $4^2=7^2+3^2-21\times 2=16$.

Also the sum multiplied by the difference of the same numbers is $10\times 4=7^2-3^2=40$.

In this illustration of the arithmetical application of the foregoing general results, *a* is interpreted by 7, and *b* by 3, but *a* and *b* may be replaced by any numbers whatever.

(2)

$$\begin{array}{llll}
a^2 & +ab & +b^2 & \\
a & -b & & \\
\hline
a^3 & +a^2b & +ab^2 & \\
 & -a^2b & -ab^2 & -b^3 \\
\hline
a^3 & & & -b^3 \\
\hline
\end{array}$$

(3)

$$\begin{array}{rrrr}
3x^2 & -\ 4x & & \\
2x & -\ 3 & & \\
\hline
6x^3 & -\ 8x^2 & & \\
 & -\ 9x^2 & +12x & \\
\hline
6x^3 & -17x^2 & +12x & \\
\hline
\end{array}$$

(4)

$$\begin{array}{rrrr}
5x^2 & -\ 2x & +3 & \\
3x & +\ 4 & & \\
\hline
15x^3 & -\ 6x^2 & +9x & \\
 & 20x^2 & -8x & +12 \\
\hline
15x^3 & +14x^2 & +\ x & +12 \\
\hline
\end{array}$$

(5)

$$\begin{array}{lll}
x & +a & \\
x & +b & \\
\hline
x^2 & +ax & \\
 & bx & +ab \\
\hline
x^2 & +(a+b)x & +ab \\
\hline
\end{array}$$

(6)

$$\begin{array}{llll}
ax^2 & +\ bx & +c & \\
3x & -b & & \\
\hline
3ax^3 & +3bx^2 & +3cx & \\
 & -abx^2 & -b^2x & -bc \\
\hline
3ax^3 & +(3b-ab)x^2 & +(3c-b^2)x & -bc \\
\hline
\end{array}$$

(7)

$$\begin{array}{rrrrrr}
12ax & -\ 5ab & +3 & & & \\
ax & -\ 2ab & +2 & & & \\
\hline
12a^2x^2 & -\ 5a^2bx & +3ax & & & \\
 & -24a^2bx & & +10a^2b^2 & -\ 6ab & \\
 & & 24ax & & -10ab & +6 \\
\hline
12a^2x^2 & -29a^2bx & +27ax & +10a^2b^2 & -16ab & +6 \\
\hline
\end{array}$$

Examples for Exercise.

(1) Multiply $3ax-4$ by $2ax+2$

(2) Multiply $12ax^2-3bx+c$ by $4x^2-3x$

(3) Multiply $8x^2y-3xy+12$ by $2x^2-4$

(4) Multiply $(3x-2)\ (2x+3)$ by $3ax-1$

(5) Multiply $(3-4x)\ (7x+5)$ by $5x+3$

(6) Multiply the square of $x+2y$ by $x-2y$

(7) What is the cube of $a+x$?

(8) Find the cube of $a-x$

(9) Multiply $x^2+\frac{1}{2}x-2x$ by 6

(10) Multiply $2x^2-\frac{1}{2}x+\frac{1}{3}$ by $4x-\frac{1}{2}$

(11) Multiply the square of $\frac{1}{2}x-\frac{1}{3}y$ by $x-y$

(12) Find the product of the factors $2x-4$, $7x+3$, $\frac{1}{2}x-2$

(13) $(3ax-4by-2)\,(5ax-2by-3)=$

(14) $(3x-4)\,(2x+\frac{1}{2})\,(7x-3)=$

(15) $(2ax+b)^2\,(ax-b)\,(ax+b)=$

(16) $(7x^3-4x^2-3x+1)\,(2x^2-3x-2)=$

(17) $\left\{x^2-(a+b)x+c\right\}\,\left\{x^2+(a+b)x-c\right\}=$

(18) Prove that $(x^3+x^2y+xy^2+y^3)\,(x-y)=(x^2+y^2)\,(x^2-y^2)$

(19) Prove that $\left\{(a^2+x^2)^2-a^2x^2\right\}\,(a^2-x^2)=(a^3+x^3)\,(a^3-x^3)$

(20) Find the result of $(x^2+y^2)\,(x^2-y^2)\,(x^4+y^4)$ without actual multiplication, by aid of the principle 3, at p. 27.

DIVISION.

15. Division of algebra, like multiplication, may be conveniently separated into three cases.

CASE I. *When dividend and divisor are both simple quantities.*

RULE 1. Write down the *sign* of the quotient; if the signs of dividend and divisor are like, it is *plus*; if unlike, it is *minus*.

2. Then write the quotient of the coefficients found by common arithmetic.

3. Lastly, annex the quotient of the letters; these being such that if they were united to the letters in the divisor they would make up the letters in the dividend.

As the correct quotient is always such that when multiplied by the divisor the product is the dividend, it is obvious that the quantity determined by these precepts will be that quotient; for when it is multiplied by the divisor, the sign, coefficient, and letters of the dividend are necessarily reproduced.

NOTE. When any letter or *factor* occurs in the divisor that is not also in the dividend, then, just as in common arithmetic, after the quotient by the other factors has been obtained, the factor unemployed must be written under it as a divisor, to imply that division by it has not been performed. This is only what we do in arithmetic, to which, indeed, the operations of algebra always correspond; thus, if we have to divide 180 by 24; then, the factors of 180 being 12, 5, and 3, and those of 24, 12 and 2, we see that after the division of 180 by 12, giving the quotient 15, we must write the unemployed divisor 2 under this quotient as a divisor, so that $\frac{15}{2}$ or $7\frac{1}{2}$ is the final quotient.

1. Divide $-8a^4x^2$ by $4a^2x$. Here the sign of the quotient is *minus*, the signs of dividend and divisor being unlike; the coefficient of the quotient is, therefore -2. As to the letters of the quotient, we have only to annex those which, if united to the letters a^2x in the divisor, would make up the letters a^4x^2 in the dividend; it is plain, therefore, that two a's, that is a^2 and one x, are wanted for this purpose; so that we have

$$\frac{-8a^4x^2}{4a^2x}=-2a^2x$$

(2) Divide $-16a^5b^4x^2y^3$ by $-4a^2bx^2y$

$$\frac{-16a^5b^4x^2y^3}{-4a^2bx^2y}=4a^3b^3y^2$$

(3) Divide $12ab^6x^4y^5$ by $-3abx^3yz$

$$\frac{12ab^6x^4y^5}{-3abx^3yz}=-\frac{4b^5xy^4}{z},\text{ or }\frac{-4b^5xy^4}{z},\text{ or }\frac{4b^5xy^4}{-z}$$

These three forms of the expression are all the same in meaning; the entire fraction is *minus*, because the dividend and divisor have unlike signs; the first way of placing the minus, directly in front of the fraction, is that most generally adopted.

(4) Divide $17a^6b^7c^8x^3y^2$ by $3a^3b^4c^2x^4y^6$

$$\frac{17a^6b^7c^8x^3y^2}{3a^3b^4c^2x^4y^6}=\frac{17a^3b^3c^6}{3xy^4}$$

You see that when powers of the same letter occur both in dividend and divisor, the quotient, in reference to that letter, is got by simply subtracting the lower exponent from the upper; thus, $\frac{a^6}{a^3}=a^3$, $\frac{b^7}{b^4}=b^3$, and so on; if, however, the lower exponent be the greater, then we can subtract only a portion of the lower from the upper, and can thus only partially perform the division; the lower letter, with its exponent thus diminished, still remains as a divisor; thus, $\frac{x^3}{x^4}=\frac{1}{x}$, $\frac{y^2}{y^6}=\frac{1}{y^4}$, and so on; for x^3 divided by x^4 is, of course, the same as x^3 divided by x^3, and then the quotient by x; but x^3 divided by x^3 is 1, and this divided by x is $\frac{1}{x}$; in like manner, y^2 divided by y^6 is the same as y^2 divided by y^2, and then the quotient by y^4; but y^2 divided by y^2 is 1, and this divided by y^4 is $\frac{1}{y^4}$.

It is very important always to remember that when powers of the same quantity are to be operated with, multiplication requires only the *addition* of their exponents, and division only the *subtraction* of their exponents.

Examples for Exercise.

(1) Divide $14ax^2y^3$ by $-2axy^2$

(2) $-a^4y^6z^2 \div 3a^3y^2x^3$

(3) $-12b^7x^4y \div -6a^3b^5x^4y^2$

(4) $5ax^3y^5 \div 3a^2xyz^4$

(5) $16a^4x^2y^{\frac{1}{2}} \div -4a^3x^3\sqrt{y}$

(6) $-7xy^4\sqrt[3]{z} \div 6x^2y^4z^{\frac{1}{3}}$

(7) $3a^2m^3xy \div 5a^4n^2y^3$

(8) $2b^7x^4z^{\frac{1}{3}} \div -\frac{1}{2}x^5yz^{\frac{1}{4}}$

(9) $-7x^{\frac{1}{2}}yz^{\frac{1}{3}} \div -2\sqrt[3]{z}$

(10) $\frac{1}{4}abc^2\sqrt{x} \div \frac{1}{2}a^2b^2cx^{\frac{1}{4}}$

(11) $9a^2n^2\sqrt{x+y} \div 18a^3n(x-y)^{\frac{1}{2}}$

(12) $3ab(x-y)^{\frac{1}{3}} \div 6a^2b\sqrt[3]{x-y}$

CASE II. *When the dividend is a compound quantity and the divisor a simple quantity.*

16. RULE 1. By the former rule find the quotient of the divisor and *each* term of the dividend.

2. Connect these several partial quotients by their proper signs and the complete quotient will be exhibited.

(1)

$$\frac{6a^2b^3x^4-4ab^2x^3+2bx^2}{2bx^2}=3a^2bx^2-2abx+1$$

(2)

$$\frac{12ax^2z^3-8a^2xz^2-4xz}{4xz^2}=3axz-2a^2-\frac{1}{z}$$

Examples for Exercise.

(1) $\dfrac{8a^2x^3-12ax^2}{2ax}$

(2) $\dfrac{12m^4x^6-8m^3x^5+16m^2x^4}{4m^2x^3}$

(3) $\dfrac{18c^5x^4y^8-3c^3x^3y^5-6c^2x^2y^3}{3c^2xy^2}$

(4) $\dfrac{5a^5b^6x^7+10a^4b^7x^6-2}{5a^4b^3x^2}$

(5) $\dfrac{3x^2y^3z^2-4x^2y^2z^3-xyz^4}{5x^2y^2z}$

(6) $\dfrac{12x^5y^7-3(x^4+2x^3y^2)^*}{3x^2y}$

(7) $\dfrac{15-9(a^3x^4+b^2x^3y^3-3)}{5x^2y^3}$

(8) $\dfrac{4(a^4x^6-2a^3b)x^2-8(a^5-3x^3)bx^2}{4abx}$

(9) $\dfrac{5(3a^2x^5-2)(7a^5x^2+3ax)-10a^3x^3}{2a^2x^2}$

(10) Divide $\{a^3-2(a^2-x^2)\}\{a^4-3(x^2-a^4)\}$ by $2ax^2$

CASE III. *When dividend and divisor are both compound quantities.*

17. RULE 1. Arrange the terms both of dividend and divisor so that the powers of some letter common to both may follow each other in

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* It is in general better to remove vincula before attempting the division; first, however, cancelling such factors as are obviously common both to dividend and divisor. In example 6 it is plain that the factor 3 is common to both, and in example 8 that $4x$ is common to both.

ascending or descending order. If, for instance, x be the letter in reference to which the arrangement of the terms is to be governed, write the term containing the highest power of x first, then the term containing the next highest power, and so on; or else write the term containing the lowest power of x first, then that containing the next lowest, and so on.

2. After having thus arranged the terms of dividend and divisor, place them just as dividend and divisor are placed in long division of common arithmetic; divide the leading term of the former by that of the latter, you will thus get the leading term of the quotient; multiply the whole of the divisor by it, and, as in arithmetic, subtract the product from the like terms of the dividend, bringing down a new term, or two terms, if one be insufficient, and you will thus have a new dividend.

3. Divide in like manner the leading term of this by the leading term of the divisor, you will thus get the second term of the quotient, which, multiplied by the divisor and the product subtracted, will give, when an additional term is brought down, a third dividend.

4. Proceed in this way till all the terms in the dividend have been brought down, and if there be any final remainder, then, as in arithmetic, this remainder, with the divisor written underneath, is the fraction to be connected with the quotient to render it complete, as in division of numbers.

(1) Divide $6x^2+13x+6$ by $3x+2$

Here the quantities are already properly arranged for the operation; the terms regularly proceed according to the descending powers of x; the work by the rule is as follows:—

$$\begin{array}{l}
3x+2\,)\,6x^2+13x+6\,(\,2x+3 \\
\qquad\quad\;\; 6x^2+\;\,4x \\
\qquad\quad\;\; \overline{\qquad\qquad\;\;} \\
\qquad\qquad\qquad 9x+6 \\
\qquad\qquad\qquad 9x+6 \\
\qquad\qquad\qquad \overline{\qquad\;\;}
\end{array}$$

As no remainder is left, the complete quotient is $2x+3$, without any fractional correction.

(2) Divide $12x^5-13x^4-34x^3+40x^2$ by $4x^2-7x$

As it is obvious here that x is a factor common to both dividend and divisor, it had better be cancelled at the outset, as it is useless to retain it; the work will then be as follows:—

$$\begin{array}{l}
4x-7\,)\,12x^4-13x^3-34x^2+40x\,(\,3x^3+2x^2-5x+\dfrac{5x}{4x-7} \\
\qquad\quad\; 12x^4-21x^3 \\
\qquad\quad\; \overline{\qquad\qquad\quad} \\
\qquad\qquad\qquad 8x^3-34x^2 \\
\qquad\qquad\qquad 8x^3-14x^2 \\
\qquad\qquad\qquad \overline{\qquad\qquad} \\
\qquad\qquad\qquad\quad -20x^2+40x \\
\qquad\qquad\qquad\quad -20x^2+35x \\
\qquad\qquad\qquad\quad \overline{\qquad\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad 5x \text{ remainder.} \\
\qquad\qquad\qquad\qquad\qquad \overline{\quad\;}
\end{array}$$

Here the fractional correction is $\frac{5x}{4x-7}$; which is united to the integral part of the quotient to render it complete.

(3) Divide $6x^4-96$ by $3x-6$. Suppressing the factor 3 common to both, the work is this—

$$\begin{array}{llllll}
x-2\,) & 2x^4-32\,(& 2x^3+4x^2+8x+16 & & & \\
 & 2x^4-4x^3 & & & & \\ \cline{2-2}
 & & 4x^3 & & & \\
 & & 4x^3-8x^2 & & & \\ \cline{3-3}
 & & & 8x^2 & & \\
 & & & 8x^2-16x & & \\ \cline{4-4}
 & & & & 16x-32 & \\
 & & & & 16x-32 & \\ \cline{5-5}
\end{array}$$

The dividend $2x^4-32$ in this example is obviously the same as $2x^4+0x^3+0x^2+0x-32$, in which a complete form is given to the expression, by filling up the gap with zero-terms. In the foregoing operation, these zero-terms, or *nothings*, as they are of no effect, are not brought down, and the bringing down of the final term 32 is postponed till it is wanted.

(4) Divide 1 by $1+x$

$$\begin{array}{llllll}
1+x\,) & 1\quad (1-x+x^2-x^3+\&c. & & & & \\
 & 1+x & & & & \\ \cline{2-2}
 & \quad -x & & & & \\
 & \quad -x-x^2 & & & & \\ \cline{2-2}
 & & x^2 & & & \\
 & & x^2+x^3 & & & \\ \cline{3-3}
 & & & -x^3 & & \\
 & & & -x^3-x^4 & & \\ \cline{4-4}
 & & & & x^4,\ \&c. & \\ \cline{5-5}
\end{array}$$

It is plain that the quotient in this example will never terminate of itself; the law of the terms is easily discovered after a few of them are found, so that we may write as many of them as we please without prolonging the work, thus—

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\&c.\ \textit{ad infinitum}\ldots.(1)$$

If we stop at the third term, the complete quotient will be

$$\frac{1}{1+x}=1-x+x^2-\frac{x^3}{1+x}$$

if at the fourth, $\frac{1}{1+x}=1-x+x^2-x^3+\frac{x^4}{1+x}$; and so on.

Let x stand for 1; then $\frac{1}{1+1}$, that is,

$$\frac{1}{2}=1-1+1-\frac{1}{2}$$
$$=1-1+1-1+\frac{1}{2}$$
$$=1-1+1-1+1-\frac{1}{2}, \text{ \&c.}$$

as is sufficiently obvious, and stop wherever we may, the *development*, as it is called, of $\frac{1}{1+1}$, that is, of $\frac{1}{2}$, cannot be complete till the fractional correction $-\frac{1}{2}$ or $+\frac{1}{2}$ is annexed. If, however, x be less than 1, then the fractional correction becoming more and more insignificant, and at length indefinitely small, and in fact 0, when the number of terms is infinite, we conclude that the development is accurately expressed by the infinite series (1) without any fractional correction; thus, if $x=\frac{1}{2}$, then $\frac{1}{1+\frac{1}{2}}$, that is,

$$\frac{2}{3}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+ \text{ \&c. } ad\ inf.$$

And we know from arithmetic that this is nothing but a *geometric* series whose common ratio is $-\frac{1}{2}$, and sum $\frac{2}{3}$.

Examples for Exercise.

(1) Divide $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$

(2) Divide a^3+x^3 by $a+x$, and also by $a-x$

(3) Divide $x^5-5ax^4+10a^2x^3-10a^3x^2+5a^4x-a^5$ by $x^2-2ax+a^2$

(4) Divide $48z^3-96az^2-64a^2z+150a^3$ by $2z-3a$

(5) Divide $9x^6-46x^5+95x^3+150x$ by x^2-4x-5

(6) Divide $x^6-x^4+x^3-x^2-1$ by x^2+x-1

(7) Divide $4(x^4-2x^2+1)$ by $2(x+1)(x-1)$

(8) Divide $\frac{1}{3}x^3+\frac{11}{6}x^2-\frac{7}{3}x+\frac{4}{3}$ by $x^2-\frac{1}{2}x+\frac{2}{3}$

(9) Divide x^3-5x^2+7x+3 by $x-2$, and show that the remainder is the same number as the dividend when the x in it is replaced by 2.

(10) Divide x^3+bx^2+cx+d by $x-a$, and show that the remainder is the same as the dividend when the x in it is replaced by a.

On the management of Exponents.

18. Positive integral exponents denote powers of the quantity which they are attached; positive fractional exponents, the

of each fraction being 1—indicate *roots*. This has already been explained (Art. 11). But negative exponents, whether integral or fractional, and without limitation as to the value of the numerator in the latter case, are also employed in algebra. It is necessary to explain the meaning of these exponents, and to show how operations with quantities with which they are connected are performed.

Negative Exponents.—You have seen (page 29) that division of powers of the same quantity is effected by simply subtracting the exponent of the divisor from that of the dividend, thus :—

$$\frac{a^6}{a^3}=a^{6-3}=a^3;\ \frac{a^3}{a^2}=a^{3-2}=a^1 \text{ or } a \text{ \&c.}$$

The following extension of the notation for powers is obviously suggested from this :—

$$\frac{a^1}{a^1}=a^{1-1}=a^0;\ \text{but } \frac{a^1}{a^1}=1\ \therefore\ a^0=1$$

$$\frac{a^1}{a^2}=a^{1-2}=a^{-1};\ \text{but } \frac{a^1}{a^2}=\frac{1}{a}\ \therefore\ a^{-1}=\frac{1}{a}$$

$$\frac{a^1}{a^3}=a^{1-3}=a^{-2};\ \text{but } \frac{a^1}{a^3}=\frac{1}{a^2}\ \therefore\ a^{-2}=\frac{1}{a^2}$$

$$\frac{a^1}{a^{n+1}}=a^{1-n-1}=a^{-n};\ \text{but } \frac{a^1}{a^{n+1}}=\frac{1}{a^n}\ \therefore\ a^{-n}=\frac{1}{a^n}$$

We thus see that whatever whole number n may represent, a^{-n} is merely another way of writing $\frac{1}{a^n}$; 1 divided by any quantity is called the *reciprocal* of that quantity; therefore, not only powers but the reciprocals of powers may be indicated by exponents, the former by positive, the latter by negative exponents.

It is also worthy of notice that, whatever a may be, a^0 is always the representative of 1; for it is only another form of $\frac{a}{a}$.

Fractional Exponents.—If we have to express the cube root of the square of a, we might employ the following notation, namely $(a^2)^{\frac{1}{3}}$, but this is unnecessarily complicated; if it be only laid down as a principle that the numerator of an exponent shall mark the power and the denominator the root of that power, then, instead of the above, we have the more compact form $a^{\frac{2}{3}}$. In like manner, instead of $(a^3)^{\frac{1}{5}}$, we have $a^{\frac{3}{5}}$, meaning the fifth root of the third power of a. And generally, instead of $(a^m)^{\frac{1}{n}}$, we have $a^{\frac{m}{n}}$, meaning the nth root of the mth power of a. It is matter of indifference whether you say the nth root of the mth power, or the mth power of the nth root; because, whether you first take the power of a number and then get the root, or first find the root and then get the power, the result is the same; that is to say, $(a^m)^{\frac{1}{n}}$, and $(a^{\frac{1}{n}})^m$ are identical as to value, whatever numbers a, m, and n may be. For example, suppose $m=3$ and $n=4$, then we have

$$(a^3)^{\frac{1}{4}}=(aaa)^{\frac{1}{4}}=a^{\frac{1}{4}}a^{\frac{1}{4}}a^{\frac{1}{4}}$$

and $$(a^{\frac{1}{4}})^3=a^{\frac{1}{4}}a^{\frac{1}{4}}a^{\frac{1}{4}}$$

so that the two forms are identical.

From what has now been said, it appears that $a^{\frac{m}{m}}$ may always be replaced by a^1 or a, because the mth root of the mth power of a is, of course, a itself; the two operations neutralize one another, and leave a unaltered. Such an expression as $a^{\frac{p}{q}\cdot\frac{m}{m}}$, or, $a^{\frac{pm}{qm}}$ is, therefore, simply $a^{\frac{p}{q}}$; for the former is but the mth root of the mth power of this, which is left unaltered by the two neutralizing operations. Whenever, therefore, a fractional exponent has a factor common to numerator and denominator, that factor may be cancelled, and the exponent be thus simplified.

All that has been stated in reference to positive exponents applies equally to negative exponents; for, since a quantity with a negative exponent is only the reciprocal of that quantity with the exponent made positive, that is, since a^{-m} is $\frac{1}{a^m}$, we have only to apply what has been said to the denominator in this reciprocal, and to change the notation of the results back again to negative exponents.

The principal operations with exponents, when attached to the same quantity, are here exhibited.

1. $a^m\times a^n=a^{m+n}$; thus, $a^3\times a^5=a^8$; $a^{\frac{1}{2}}\times a^{\frac{2}{3}}=a^{\frac{1}{2}+\frac{2}{3}}=a^{\frac{7}{6}}$

2. $a^m\div a^n=a^{m-n}$; thus, $a^3\div a^5=a^{3-5}=a^{-2}=\frac{1}{a^2}$;

$$a^{\frac{1}{2}}\div a^{\frac{2}{3}}=a^{\frac{1}{2}-\frac{2}{3}}=a^{-\frac{1}{6}}=\frac{1}{a^{\frac{1}{6}}}=\frac{1}{\sqrt[6]{a}}$$

3. $(a^m)^n=a^{mn}$; thus, $(a^3)^5=a^{15}$; $(a^{\frac{1}{2}})^{\frac{2}{3}}=a^{\frac{1}{2}\cdot\frac{2}{3}}=a^{\frac{1}{3}}=\sqrt[3]{a}$

These three general principles must be borne in remembrance, as they are of very frequent application in algebra. They may be expressed in words as follows :—

For Multiplication.—Add the exponents.

Division.—Subtract exponent of divisor from exponent of dividend.

Powers or Roots.—Multiply the exponent of the quantity whose power or root is to be found by the exponent of that power or root.

(1) Express the product $a^{\frac{3}{4}}\times a^{\frac{1}{3}}$; and $a^{-\frac{1}{2}}\times a^4$

(2) Express the quotient $a^{\frac{3}{4}}\div a^{\frac{1}{3}}$; and $a^{-\frac{1}{2}}\div a^4$

(3) What is the value of $(a^{\frac{1}{3}})^{\frac{1}{4}}$; and $(a^{-\frac{1}{2}})^{\frac{3}{4}}$?

The work of these six examples is as follows :—

(1) $a^{\frac{3}{4}}\times a^{\frac{1}{3}}=a^{\frac{3}{4}+\frac{1}{3}}=a^{\frac{13}{12}}=a\cdot a^{\frac{1}{12}}$; $a^{-\frac{1}{2}}\times a^4=a^{4-\frac{1}{2}}=a^{\frac{7}{2}}=a^3\sqrt{a}$

(2) $a^{\frac{3}{4}}\div a^{\frac{1}{3}}=a^{\frac{3}{4}-\frac{1}{3}}=a^{\frac{5}{12}}$; $a^{-\frac{1}{2}}\div a^4=a^{-\frac{1}{2}-4}=a^{-\frac{9}{2}}=\frac{1}{a^{\frac{9}{2}}}=\frac{1}{\sqrt{a^9}}$

(3) $(a^{\frac{1}{3}})^{\frac{1}{4}}=a^{\frac{1}{12}}$; $(a^{-\frac{1}{2}})^{\frac{3}{4}}=a^{-\frac{3}{8}}=\frac{1}{a^{\frac{3}{8}}}=\frac{1}{\sqrt[8]{a^3}}$

19. In extracting roots, or in *evolution*, as it is called, some caution must be exercised as to *signs*. In the operation of which this is the reverse, that is, in *involution*, or the raising of powers, the rule of signs guides us at once to the correct sign of the result, which sign is never attended with any ambiguity, for an even number of minus factors and any number of plus factors always gives plus, and an odd number of minus factors always gives minus. But in the reverse operation of extracting a root, the sign is often ambiguous; thus, the numerical value of $\sqrt{4}$ is 2; but we have no right to say that this 2 is plus any more than minus, since 2^2 and $(-2)^2$ are equally 4. Seeing, therefore, that there is nothing to warrant any preference as to sign, we say that the square root of 4 is ambiguously plus or minus, or in symbols, that $\sqrt{4}=\pm 2$. Every even root of a positive quantity is, in like manner, ambiguous as to sign; thus, $\sqrt{16}=\pm 4$, $\sqrt{9}=\pm 3$, $\sqrt[4]{16}=\pm 2$, &c. An even root of a *negative* quantity is impossible; for no even number of factors, whether they be all positive or all negative, can ever give a negative result; and you know *that* only is a root of a quantity which, upon being raised to the corresponding power, reproduces that quantity. Such an expression as $\sqrt{-1}$, or $\sqrt{-4}$, or $\sqrt{-9}$, &c., implies an operation of impossible performance; for there does not exist any number which, multiplied by itself, or squared, can produce -1, or -4, or -9, or indeed *minus* anything. Expressions of the kind here noticed are hence called *impossible*, or *imaginary* quantities. Whenever they occur in the result of any algebraical inquiry, they indicate that the conditions of the problem are impossible; whenever the problem can be solved for real values of the unknown quantity, algebra will furnish those values; when such is not the case, the entrance of an imaginary quantity into the expression for the sought value will apprise us of the fact.

If, however, an even root of a negative quantity be accompanied by the corresponding even power, then, as the two neutralize each other, the negative quantity itself is the result; for this negative quantity, thus submitted to operations that counteract each other, remains virtually unoperated upon at all; thus $(\sqrt{-4})^2=-4$, $(\sqrt{-6})^2=-6$, and so on. It is necessary that you should always remember that the square of a square root of anything is the thing itself, and so is the cube of the cube root, the fourth power of the fourth root, and generally the nth power of the nth root.

An algebraical result may involve a root that cannot be accurately extracted; such roots are $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt[3]{9}$, &c.; but we know from arithmetic that, by aid of decimals, we can *approximate* to any such root to a degree of nearness as close as we please; such expressions are called *surds*, or *irrational* quantities; expressions indicating no extraction, or in which the extraction, if indicated, can be accurately executed, being called *rational* expressions.

It may be as well to give here more formal definitions of the new terms introduced into the present article.

DEFINITIONS.

20. *Involution*: the raising of powers.

Evolution: the extracting of roots.

Imaginary or impossible quantity: any even root of a negative number.

Surd or irrational quantity: a root that cannot be accurately extracted, but which may be approximated to.

Rational quantity: a quantity unaffected with a radical sign, or of which, when encumbered with a radical, the root can be accurately extracted.

NOTE. A quantity consisting of but one term is frequently called a *monomial*, one of two terms a *binomial*, one of three terms a *trinomial*, of four a *quadrinomial*. Quantities consisting of several terms, without specifying their number, are called *polynomials* or *multinomials*.

To extract any proposed root of a monomial.

21. RULE 1. First find the root of the coefficient, remembering that if the root be odd, then—the same as for an odd power—the sign is that of the proposed quantity; and that an even root of a positive quantity is *plus* or *minus* (see Ex. 1, below).

2. Find the root of the literal part of the term, by dividing the exponent of each letter by the index of the radical sign, or, which is the same thing, by multiplying each exponent by the fractional index that replaces the radical.

NOTE. If the coefficient be a surd, in reference to the proposed root, then you can do no more than indicate the required extraction. It often happens, however, that the coefficient may be split into two factors, one of which may admit of the indicated extraction, in which case, the radical sign, or the equivalent exponent, is to remain attached only to the other factor, as in Examples 3, 5, &c., following :—

(1) $\sqrt{}4a^6x^4=\pm 2a^3x^2$. (2) $\sqrt[3]{}8a^6x^3=2a^2x$

(3) $\sqrt{}12a^2x^4y^6=\sqrt{}(4\cdot 3a^2x^4y^6)=2\sqrt{}3\cdot ax^2y^3=2ax^2y^3\sqrt{}3$

In this third example, and in all similar instances, it is better to put the factor under the radical *last*, because then no ambiguity can arise as to how far the influence of the radical extends. You see in the first form it was necessary, to prevent the radical from applying to all that follows it, to interpose a dot; this, however, might have been dispensed with if an exponent had been used instead of the radical sign thus, $2(3^{\frac{1}{2}}ax^2y^3)$.

(4) $\sqrt[3]{}-27a^3b^2x^2=-3a\sqrt[3]{}b^2x^2=-3ab^{\frac{2}{3}}x^{\frac{2}{3}}$

(5) $\sqrt{}72a^3b^4x^8=\sqrt{}(36\cdot 2)a^{\frac{3}{2}}b^2x^4=6ab^2x^4\sqrt{}2a$

The double sign is omitted in the result of this example, as also in that of example 3, because the sign $\sqrt{}$ still enters, and therefore virtually the double sign still enters ; if a, in example 5, were 2, then $\sqrt{}2a$ would be ± 2, and the result would be $\pm 12ab^2x^4$; it is only *after* $\sqrt{}$ no longer enters, in consequence of the indicated operation being performed, that the double sign need be prefixed to the result.

(6) $\dfrac{1}{\sqrt{a^5b^3}}=\dfrac{1}{\sqrt{a^4b^2ab}}=\dfrac{1}{a^2b\sqrt{ab}}$ or $a^{-2}b^{-1}a^{-\frac{1}{2}}b^{-\frac{1}{2}}$

It is in general better to remove radical signs from *denominators* o fractions, and to transfer them to the numerators; this may be done b aid of the well-known principle in arithmetic, that the terms of a fraction may be each multiplied by any factor without disturbing the valu of the fraction; thus, by multiplying numerator and denominator o $\frac{1}{a^2b\sqrt{ab}}$ by $\sqrt{ab}$, the form is changed to $\frac{\sqrt{ab}}{a^2bab}$, that is, to $\frac{\sqrt{ab}}{a^3b^2}$; whic is a better form than the preceding; you will perceive that it is bette by reflecting that when the numerical value of $\sqrt{ab}$ is found—whic value will probably involve a string of decimals—it is easier to divide it by the value of a^3b^2 than first to multiply it by the value of a^2b, and then to divide 1 by the product.

$$(7)\ \sqrt{\frac{8x^3}{75a^2y^5}}=\sqrt{\frac{4\cdot2\cdot x^3}{25\cdot3a^2y^5}}=\frac{2}{5a}\sqrt{\frac{2x^3}{3y^5}}=\frac{2x}{5ay^2}\sqrt{\frac{2x}{3y}}=$$
$$\frac{2x}{5ay^2}\cdot\frac{\sqrt{6xy}}{3y}=\frac{2x}{15ay^3}\sqrt{6xy}$$

Examples for Exercise.

(1) $\sqrt{a^2x^6y^8}$ (2) $\sqrt{9a^4x^2y}$ (3) $\sqrt{12a^2x^3y}$ (4) $\sqrt{\frac{a^2}{b^3}}$

(5) $\sqrt[3]{a^4x^4y^6}$ (6) $\sqrt[3]{8a^3x^4y^2}$ (7) $\sqrt[3]{27b^4c^5x^6}\cdot\sqrt[3]{\frac{a^4}{b^5}}$

(8) $\sqrt{\frac{2x^5}{3y^6}}$ (9) $(16a^4b^2y^6)^{-\frac{1}{2}}$ (10) $a(x^2y^3)^{-\frac{1}{3}}$

(11) $(288x^5y^7)^{-\frac{1}{2}}$ (12) $(-8ax^3y^5)^{-\frac{1}{3}}$ (13) $\sqrt{\frac{63a^4x^3}{50y^5}}$

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(14) What is the third power of $\frac{1}{3}\sqrt{3}$?
(15) What is the fourth power of $\frac{1}{2}\sqrt{2}$?
(16) What is the square root of $16\sqrt{6}$?
(17) Required the square root of 6^3.
(18) Required the cube root of $\frac{1}{8}a^3b$, and the mth root of $x^{\frac{1}{n}}$.

FRACTIONS.

22. The rules for the treatment of numerical fractions in commo arithmetic sufficiently suggest the methods of proceeding when th fractions are those of algebra; the operations are exactly the same fo both, the only difference being in the symbols operated upon.

To reduce a mixed quantity to an improper fraction.

Rule. As in arithmetic, multiply the integral part by the denominator of the fractional part; to the product, unite the numerator wit its sign, and place the denominator underneath, thus:—

(1) $a+\frac{2x}{b}=\frac{ab+2x}{b}$ (2) $3a-\frac{5}{6}=\frac{18a-5}{6}$ (3) $a+x-\frac{b}{a-x}=\frac{a^2-x^2-}{a-x}$

(4) $a-b-\frac{m}{x}=\frac{(a-b)x-m}{x}$ (5) $\sqrt{x}-\frac{a-b}{\sqrt{x}}=\frac{x-a+b}{\sqrt{x}}$

(6) $\sqrt{x}+\frac{a-x}{\sqrt{x}}=\frac{a}{\sqrt{x}}$ (7) $3-x-\frac{2-x^2}{3+x}=\frac{7}{3+x}$

Examples for Exercise.

(1) Reduce $1-\frac{3a}{x}$ to an improper fraction.

(2) Reduce $2ab-\frac{1-3abc+m}{2c}$ to an improper fraction.

(3) Reduce $4+2x-\frac{2x^3-3a}{5a}$ to an improper fraction.

(4) Reduce $x^2-7x+10+\frac{x-5}{4x+3}$ to an improper fraction.

(5) Reduce $x^2-a^2+\frac{x^4+a^4}{x^2+a^2}$ to an improper fraction.

(6) Reduce $a^2+ax+x^2+\frac{2x^3}{a-x}$ to an improper fraction.

(7) Reduce $\frac{(a+b+c)(a+b-c)}{2ab}-1$ to an improper fraction.

(8) Reduce $\frac{a^2+b^2}{ab}+2$, and $\frac{a+b^2}{ab}-2$ to improper fractions.

To reduce an improper fraction to a whole or mixed quantity.

23. **Rule.** Actually perform the division of numerator by denominator, and to the quotient annex the remainder with the denominator underneath.

Note. This operation is nothing more than that of common division, the quotient being completed, when there is a final remainder, by annexing to it the fractional correction, as in the examples at page 31.

(1) $\frac{9ax^2+6ax+2}{3ax}=3x+2+\frac{2}{3ax}$ (2) $\frac{3ax+4x^2}{a+x}=3x+\frac{x^2}{a+x}$

Examples for Exercise.

Reduce the following to whole or mixed quantities.

(1) $\frac{a^2-b^2+c}{a-b}$ (2) $\frac{x^6-y^6-c}{x^3-y^3}$ (3) $\frac{2(x^3-y^3)}{x-y}$

(4) $\frac{8a^2-4a-7}{4a}$ (5) $\frac{2x^4}{x^2+a^2}$ (6) $\frac{15a^3+5a^2}{3a^3+2a^2-2a-4}$

(7) $\frac{x^3-y^3}{x^2+xy+y^2}$ (8) $\frac{x^4-8x+7}{x^2-3x+2}$ (9) $\frac{(a+x)^2+(a-x)^2}{a+2x}$

To reduce fractions to a common denominator.

24. RULE 1. Multiply each numerator by the product of all the denominators except its own; you will thus get the numerators of the changed fractions.

2. Multiply *all* the denominators together; the product will be the denominator common to all the changed fractions.

NOTE. It will be seen that, by proceeding in this way, the terms of each of the given fractions are both multiplied by the same thing, namely, by the product of the denominators of all the *other* fractions; and, consequently, that the resulting fractions, though changed in appearance, are not altered in value.

The above rule will always effect the required object, but sometimes the new fractions furnished by it will all admit of simplification, in consequence of the same common factor entering into both numerator and denominator of each. It is, of course, better to provide against the entrance of this superfluous factor, than to first introduce it and then expunge it. In order that the common denominator may be the least possible, it must be the lowest number or quantity that will admit of division by *each* of the given denominators; such a quantity is called *the least common multiple* of the denominators; and whenever this least common multiple is readily discoverable by a glance at the denominators, as it often is, it should be employed instead of the less simple form given by the rule. Your experience of the application of this principle, in the fractions of common arithmetic, ought to render what is here said quite intelligible; the following examples, however, will sufficiently illustrate it. Observe that the letters L. C. M. stand for "least common multiple."

(1) Change the fractions $\frac{m}{x}, \frac{n}{y}, \frac{a}{z}$ to others of the same value, and having a common denominator.

By the rule, the numerators of the changed fractions are myz, nxz, axy, and the common denominator is xyz; therefore, the changed fractions are $\frac{myz}{xyz}, \frac{nxz}{xyz}, \frac{axy}{xyz}$.

(2) Reduce $\frac{3a}{4}, \frac{5b}{6}, \frac{1}{3}$ to fractions with a common denominator.

Here it is easily seen that 12 is the least common multiple of the denominators; 12 is divisible by 4, by 6, and by 3, but no number below 12 has these divisors. We conclude, therefore, to make 12 the common denominator; in order to which, the terms of the first fraction must be multiplied by 3, those of the second by 2, and those of the third by 4; the equivalent fractions are therefore $\frac{9a}{12}, \frac{10b}{12}, \frac{4}{12}$.

The fractions furnished by the rule would be $\frac{54a}{72}, \frac{60b}{72}, \frac{24}{72}$, which, though the same in value, are much less simple than the former.

(3) Reduce $\frac{5x}{x^2-a^2}, \frac{2}{x+a}$ to fractions with a common denominator.

Since $x^2-a^2=(x+a)(x-a)$, it follows that x^2-a^2 is divisible by both denominators; it is their L. C. M. The first fraction has this denominator already; it only remains, therefore, to multiply the terms of the second by $x-a$; the changed fractions are, therefore, $\frac{5x}{x^2-a^2}$, $\frac{2(x-a)}{x^2-a^2}$.

(4) Reduce $\frac{3a}{4b^2}, \frac{2x}{3b}, \frac{y}{2b}$ to fractions with a common denominator.

Here the L. C. M. is $4b^2\times 3=12b^2$; hence the fractions are

$$\frac{9a}{12b^2}, \frac{8bx}{12b^2}, \frac{6by}{12b^2}$$

Examples for Exercise.

Reduce the following fractions to equivalent ones having a common denominator.

(1) $\frac{2x}{a}, \frac{3a+x}{2b}$ (2) $\frac{a+b}{2c}, \frac{b}{a-b}$ (3) $\frac{2ab}{a+b}, \frac{1-b}{a-b}$

(4) $\frac{a}{x-y}, \frac{b}{x^2-y^2}$ (5) $\frac{a}{x+y}, \frac{b}{x-y}, \frac{c}{x^2-y^2}$ (6) $\frac{3x^2-4a}{5}, \frac{4}{7}$

(7) $\frac{ax}{1-x^2}, \frac{a}{(1-x)^2}$ (8) $\frac{a+x}{a^2-x^2}, \frac{a-x}{a+x}, \frac{1}{a-x}$ (9) $\frac{3x^2-2x+1}{x^2-4}, \frac{2x+1}{x^2-2}$

(10) $\frac{2x+3}{a}, \frac{3x-2}{b}, \frac{3}{c}$ (11) $\frac{x-1}{x+1}, \frac{x+1}{x-1}, \frac{3}{4}, 5$

Addition and Subtraction of Fractions.

25. **Rule.—For Addition.** Reduce the fractions to a common denominator; take the sum of the changed numerators and place under it the common denominator.

For Subtraction. Reduce the fractions to a common denominator; take the difference of the changed numerators and place under it the common denominator.

(1) $\frac{a+x}{3}+\frac{a-x}{2}=\frac{2a+2x+3a-3x}{6}=\frac{5a-x}{6}$

(2) $\frac{4bx}{3a}+\frac{2ax}{5b}=\frac{20b^2x+6a^2x}{15ab}=\frac{2(10b^2+3a^2)x}{15ab}$

(3) $\frac{2x^2-5}{3x}-\frac{x+4}{3}=\frac{2x^2-5-x^2-4x}{3x}=\frac{x^2-4x-5}{3x}$

(4) $\frac{x^2+a^2}{ax}-2=\frac{x^2+a^2-2ax}{ax}=\frac{(x-a)^2}{ax}$

Examples for Exercise.

(1) $\frac{a}{3}+\frac{a}{4}+\frac{a}{5}$ (2) $\frac{2b}{x^2+b^2}+\frac{1}{x}$ (3) $\frac{x+y}{x-y}+\frac{x-y}{x+y}$

(4) $2x+\frac{x-3}{4}+\frac{2x-1}{2}$ (5) $\frac{1}{x-y}-\frac{1}{x^2-y^2}$ (6) $\frac{ax}{3}+\frac{bx}{4}-\frac{3x}{5}$

(7) $\frac{5x-3}{x+1}-\frac{2x^2-13x+1}{x^2-1}$ (8) $\frac{x+y}{x-y}-\frac{x-y}{x+y}$ (9) $1-\frac{a^2-b^2}{a^2+b^2}$

(10) $\frac{2x-3}{3x}+\frac{4x^2+3}{3x}-\frac{4x+2}{3}$ (11) $\frac{a^2+x^2}{a^2-x^2}-\frac{a+x}{a-x}+a^2+x^2$

MULTIPLICATION AND DIVISION OF FRACTIONS.

26. **FOR MULTIPLICATION.** 1. Multiply the numerators together; the result will be the numerator of the product.

2. Multiply the denominators together; the result will be the denominator of the product.

FOR DIVISION. Invert the terms of the divisor, or turn the fraction upside down, and then proceed as in multiplication.

NOTE. These rules are sufficiently demonstrated in the author's Rudimentary Treatise on Arithmetic.

Before performing either operation, reduce the fractions to lower terms, whenever a common factor is seen to enter numerator and denominator, by cancelling that factor. A fraction is in its *lowest terms*, when there exists no factor common to numerator and denominator.

(1) $\frac{3x}{4a}\times\frac{2x}{5a}=\frac{3x^2}{10a^2}$. In this example, the factor 2 in the second numerator and the factor 2 in the first denominator may be cancelled, and the multiplication regarded as the same as $\frac{3x}{2a}\times\frac{x}{5a}$. You should always be on the look-out for simplifications of this kind.

(2) $\frac{a+b}{2a+c}\div\frac{b}{2a}=\frac{a+b}{2a+c}\times\frac{2a}{b}=\frac{2a(a+b)}{b(2a+c)}$

(3) $\frac{2a^2-2b^2}{3bc}\times\frac{4a+2b^2}{a+b}=\frac{2(a-b)}{3bc}\times 2(2a+b^2)=\frac{4(2a^2-2ab+ab^2-b^3)}{3bc}$

Examples for Exercise.

(1) $\frac{3x}{4}\times\frac{2y}{5}\times\frac{2a}{7}$ (2) $\frac{3(a+b)}{5}\times\frac{4(a-b)}{5}$ (3) $\frac{2ax}{3}\times\frac{5}{a^2x^2}\times 6$

(4) $\frac{2\sqrt{x}}{y^2}\times\frac{3y}{\sqrt{x}}$ (5) $\frac{x^3-a^3}{x+2a}\div\frac{x-a}{4x+8a}$ (6) $\left(\frac{x^2}{y^2}-\frac{a^2}{b^2}\right)\div\left(\frac{x}{y}+\frac{a}{b}\right)$

(7) $\frac{a^4-2a^2x^2+x^4}{a^3x+ax^3}\div\frac{a^2-x^2}{a^2+x^2}$ (8) $\frac{a+x}{a-x}\times\frac{a^2+x^2}{(a+x)^2}$

(9) $\frac{a+x}{a}\times\frac{a-x}{x}\times\frac{a^2-x^2}{a^2+x^2}$ (10) $\frac{a^2-x^2}{x}\times\frac{x}{a+x}\times\frac{1}{a-x}$

(11) $\frac{2a+b}{3a-2b}\div\frac{3a+2b}{4a+2b}$ (12) $\frac{3\sqrt{(a+x)}}{5}\div\frac{4(a+x)}{7}$

27. **NOTE.** In operations with fractions, the three principles stated at page 26 will be found of very frequent application. To be expert in

these operations, you must acquire the power of recognizing, at once, when an expression of three terms is the square of a binomial, and must also remember that the difference of two squares may always be replaced by the product of two factors, one factor being the sum and the other the difference of the roots of those squares. Thus, in Ex. 6 above, the dividend is the same as $\left(\frac{x}{y}+\frac{a}{b}\right)\left(\frac{x}{y}-\frac{a}{b}\right)$, and, in Ex. 7, the numerator of the dividend is $(a^2-x^2)^2$. Much algebraical work is frequently saved by taking advantage of these properties.

The ordinary operations with algebraic fractions being well understood, as also the arithmetic of exponents, the subject of simple equations may be more fully discussed; in what has already been given under that head, algebraic fractions were, from necessity, excluded.

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28. *On the solution of simple equations with one unknown quantity.*

GENERAL RULE.

1. Clear the equation of fractions, if any enter; this may be done by multiplying each numerator by all the denominators except its own, or, which is, in general, the preferable way, by finding a common multiple (the smaller the better) of all the denominators, and then multiplying each numerator by that multiple, after expunging from it the factor which is equal to the denominator.

2. Clear the equation of radical signs, if any enter; this is done by causing the radical, to be removed, to occupy one side of the equation *by itself*, all the remaining terms being placed on the other side, and then performing, on each side, the involution operation, which is the reverse of that indicated by the radical; thus, if the radical be $\sqrt{}$, *square* both sides, if it be $\sqrt[3]{}$, *cube* both sides, and so on. It is plain that this operation, on the *irrational* side, is simply equivalent to rubbing the radical out. If a second radical enter, a similar course must be followed to remove *it*, as, in general, the radicals can be cleared only one at a time.

Nothing need be added here as respects the common operations of transposition, collecting the terms, &c.; these simple matters have been sufficiently explained and illustrated at page 16.

NOTE. The most convenient order in which the general precepts in this and in the former rule may be applied in any particular case, must be suggested by the example itself; but as fractions are usually more troublesome than integral quantities, it will be advisable to remove them either at first, or at an early stage of the work.

The specimens of solution which follow will, however, afford some guidance as to the application of the rules.

(1) Given the equation $\frac{3x}{4}+\frac{2x}{3}=14-\frac{x}{2}$

Clearing fractions by precept 1 of the rule, the result is

$$18x+16x=336-12x$$

$$\text{Transposing,}\qquad 18x+16x+12x=336$$
$$\text{Collecting terms,}\qquad 46x=336$$
$$\therefore\ x=\frac{336}{46}=7\tfrac{7}{23}$$

The first step in this solution is virtually got by multiplying each side of the given equation by the product of all the denominators; and such is *always* implied in the general precept that has been followed; but as the factor 2 is common to two of the denominators, this 2 may be cancelled from the product of the denominators, and we may multiply each side of the equation by 12, the L. C. M. of the denominators; and, in proceeding from fraction to fraction with this multiplier, we suppress that factor in the 12, which is the same as the denominator, and multiply the numerator only by what is left, that is, we multiply the first numerator by 3, the second by 4, the third (regarding the 14 as $\frac{14}{1}$) by 12, and the fourth by 6; the equation, thus cleared of fractions, is,

$$9x+8x=168-6x.$$
$$\text{Transposing,}\qquad 9x+8x+6x=168$$
$$\text{Collecting terms,}\qquad 23x=168$$
$$\therefore\ x=\frac{168}{23}=7\tfrac{7}{23}$$

You perceive that, in this second solution, the steps all involve smaller numbers than the corresponding steps of the former solution. And this is the advantage of using the L. C. M. whenever any two or more of the denominators have a factor in common. Observe always to suppress that factor in the L. C. M. which is equivalent to the denominator of the fraction to which the L. C. M. is applied.

(2) Given $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$ to find x.

The L. C. M. of the denominators is evidently 6; therefore, multiplying both sides by 6, the equation, cleared of fractions, is,

$$2x-10+3x=72-2x+20$$
$$\text{Transposing,}\qquad 2x+2x+3x=72+10+20$$
$$\text{Collecting terms,}\qquad 7x=102$$
$$\therefore\ x=\frac{102}{7}=14\tfrac{4}{7}$$

(3) Given $\frac{\sqrt{3x}}{2}+7=10$ to find x.

Transposing and multiplying by 2, $\sqrt{3x}=6$

Squaring, $3x=36\ \therefore\ x=12$

(4) Given $\sqrt{(ax+x^2)}+a+x=2a$ to find x.

Transposing, agreeably to precept 2 of the rule,

$$\sqrt{(ax+x^2)}=a-x$$
$$\text{Squaring,}\qquad ax+x^2=a^2-2ax+x^2$$
$$\text{or,}\qquad ax=a^2-2ax$$
$$\text{Transposing,}\qquad 3ax=a^2\ \therefore\ x=\frac{a^2}{3a}=\frac{a}{3}$$

(5) Given $\sqrt{(4+x)}+\sqrt{x}=4$ to find x.

Transposing, $\sqrt{(4+x)}=4-\sqrt{x}$

Squaring, $4+x=16-8\sqrt{x}+x$

Transposing, $8\sqrt{x}=12 \quad \therefore \quad 2\sqrt{x}=3$

Squaring, $4x=9 \quad \therefore \quad x=2\frac{1}{4}$

(6) Given $\sqrt{x}+\sqrt{(2a+x)}=\dfrac{4a}{\sqrt{(2a+x)}}$ to find x.

Multiplying by $\sqrt{(2a+x)}$, we have

$$\sqrt{(2ax+x^2)}+2a+x=4a$$

Transposing, $\sqrt{(2ax+x^2)}=2a-x$

Squaring, $2ax+x^2=4a^2-4ax+x^2$

that is, $2ax=4a^2-4ax$

Transposing, $4ax+2ax=4a^2$

or, $2x+x=3x=2a$

$$\therefore \quad x=\frac{2a}{3}$$

(7) Given $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$ to find x.

Squaring, $a^2+2ax+x^2=a^2+x\sqrt{(b^2+x^2)}$

that is, subtracting a^2 from each side,

$$2ax+x^2=x\sqrt{(b^2+x^2)}$$

Dividing by x, $2a+x=\sqrt{(b^2+x^2)}$

Squaring, $4a^2+4ax+x^2=b^2+x^2$

Subtracting x^2 from each side,

$$4a^2+4ax=b^2$$

Transposing, $4ax=b^2-4a^2$

$$\therefore \quad x=\frac{b^2-4a^2}{4a}$$

NOTE. The following property of two equal fractions is of considerable use in the solution of a certain class of equations involving fractions.

Let the fractions $\frac{a}{b}$, $\frac{c}{d}$ be equal, then adding, or subtracting 1, we have,

$$\frac{a}{b}+1=\frac{c}{d}+1, \text{ or } \frac{a+b}{b}=\frac{c+d}{d} \ldots\ldots(1)$$

$$\frac{a}{b}-1=\frac{c}{d}-1, \text{ or } \frac{a-b}{b}=\frac{c-d}{d} \ldots\ldots(2)$$

Divide (1) by (2), and we get the property referred to; namely, that if $\frac{a}{b}=\frac{c}{d}$, then also $\frac{a+b}{a-b}=\frac{c+d}{c-d}$; that is, the quotient of the sum and difference of the terms of the one fraction, is equal to the quotient of the sum and difference of the terms of the other.

(8) Given the equation $\dfrac{\sqrt{x}+2}{\sqrt{x}-2}=\dfrac{3}{2}$ to find x.

From the principle just explained,

$$\frac{2\sqrt{x}}{4}=\frac{5}{1} \quad \therefore \quad \frac{\sqrt{x}}{2}=5 \quad \therefore \quad \sqrt{x}=10$$

and squaring, $x=100$

(9) Given $\frac{\sqrt{x}+28}{\sqrt{x}+4}=\frac{\sqrt{x}+38}{\sqrt{x}+6}$ to find x.

By the foregoing principle,

$$\frac{2\sqrt{x}+32}{24}=\frac{2\sqrt{x}+44}{32}$$

$$\therefore \quad \frac{\sqrt{x}+16}{12}=\frac{\sqrt{x}+22}{16}$$

Clearing fractions, by multiplying by 48,

$$4\sqrt{x}+64=3\sqrt{x}+66$$

Transposing, $\sqrt{x}=2 \quad \therefore \quad x=4$

(10) Given $\frac{\sqrt{(x+1)}-\sqrt{(x-1)}}{\sqrt{(x+1)}+\sqrt{(x-1)}}=\frac{1}{2}$ to find x.

Applying the principle above to these fractions, we have

$$\frac{\sqrt{(x+1)}}{\sqrt{(x-1)}}=\frac{3}{1}=3$$

Squaring, $\frac{x+1}{x-1}=9$

Clearing, $x+1=9x-9$

Transposing, $10=8x \quad \therefore \quad x=\frac{10}{8}=1\tfrac{1}{4}$

Examples for Exercise.

(1) $x-\frac{3x}{4}-9=0$ (2) $\frac{x-2}{2}+\frac{x}{3}=20-\frac{x-6}{2}$ (3) $\frac{x}{5}+\frac{x}{4}=\frac{x}{2}-\frac{x}{3}+17$

(4) $2x-\frac{x+3}{3}+15=\frac{12x+26}{5}$ (5) $4a-x=\frac{x^2}{4a-x}$ (6) $\sqrt{x}-1=7$

(7) $\sqrt{(3x-1)}=4$ (8) $1+\sqrt{x}=\sqrt{(x+7)}$ (9) $\sqrt[3]{(2x-5)}=2$

(10) $\frac{5x-9}{\sqrt{5x}+3}=\frac{\sqrt{5x}-1}{2}$ (11) $\frac{\sqrt{(1+x^2)}+x}{\sqrt{(1+x^2)}-x}=\frac{a}{b}$

(12) $x+\sqrt{(a-x)}=\frac{a}{\sqrt{(a-x)}}$ (13) $x^2+x^{-2}=(x-x^{-1})^2+x$

(14) $\frac{1}{7}\left(x-\frac{1}{2}\right)-\frac{1}{5}\left(\frac{2}{3}-x\right)=\frac{43}{30}$ (15) $\frac{\sqrt{a}+\sqrt{(a-x)}}{\sqrt{a}-\sqrt{(a-x)}}=m$

(16) $\frac{\sqrt[3]{(x+1)}-\sqrt[3]{(x-1)}}{\sqrt[3]{(x+1)}+\sqrt[3]{(x-1)}}=\frac{2}{3}$ (17) $1+x=\sqrt{\{1+x\sqrt{(16+x^2)}\}}$

(18) $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$ (19) $\sqrt{(a+x)}+\sqrt{(a-x)}-2\sqrt{x}=0$ (20) $(4\sqrt{x}-\sqrt{3})^3=81\sqrt{3}$

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29. *Questions to be solved by simple equations containing only one unknown quantity.*

(1) Find a number such that whether it is divided into two or into three equal parts, the product of the parts shall be the same.

Let x be the number, then, by the question,

$$\frac{x}{2}\times\frac{x}{2}=\frac{x}{3}\times\frac{x}{3}\times\frac{x}{3};\text{ or }\frac{x^2}{4}=\frac{x^3}{27}$$

Dividing each side by x^2,

$$\frac{1}{4}=\frac{x}{27}\quad\therefore\quad\text{clearing, }27=4x$$

$$\therefore\quad x=\frac{27}{4}=6\tfrac{3}{4}$$

(2) There are two sorts of spirits, one sort worth 20*s.* a gallon, and the other sort worth 12*s.*: how much of each must be taken to make a gallon worth 14*s.*?

Suppose $\frac{1}{x}$th part of a gallon of that at 20*s.*, then there must be the $\left(1-\frac{1}{x}\right)$th part of a gallon of that at 12*s.* The worth of the former part is $\frac{20}{x}$ shillings, and the worth of the latter part $12-\frac{12}{x}$ shillings; and by the question $\frac{20}{x}+12-\frac{12}{x}=14$; that is, $\frac{8}{x}=2$.

Multiplying by x, $\quad 8=2x \quad \therefore \quad x=4$

Consequently there must be $\frac{1}{4}$ gallon at 20*s.*, and $\therefore$ $\frac{3}{4}$ gallon at 12*s.* It is plain that $\frac{1}{4}$ of 20*s.* + $\frac{3}{4}$ of 12*s.* = 14*s.*

(3) A person bought a certain number of sheep for £94; having lost 7 of them, he sold one-fourth of the remainder for what they cost him, namely £20. How many had he at first?

(4) Find a number such that if it be increased by 11 the square root of the sum may be equal to the square root of the number itself, and 1 more.

(5) A labourer is engaged to reap 20 acres, part of which is wheat and part oats, for £6. 6*s.*; the wheat is to be paid for at the rate of 7*s.* per acre, and the oats at the rate of 5*s.* But the labourer falling ill, reaped only the wheat: how much money ought he to receive?

(6) A gamester playing at cards betted 3*s.* to 2*s.* upon every deal; after twenty deals, he had won 5*s.*: how many of the twenty did he lose?

(7) Divide the number 116 into four parts, such that if the first be increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results may all be the same.

(8) A messenger starts from a certain place and walks at the rate of $3\frac{3}{4}$ miles an hour; but half an hour after he has left, another is despatched to overtake him, which he does after walking two hours: at what rate does he walk?

(9) A merchant has spirits at 9*s.* 6*d.* per gallon, and at 13*s.* 6*d.* per gallon: how many gallons of each must he take to make 104 gallons that shall be worth £56?

(10) A cistern is supplied by three pipes, A, B, C; A can fill it in a minutes, B in b minutes, and C in c minutes: in how many minutes can it be filled if all the pipes run together?

(11) Divide the number 91 into two parts, such that if the greater part be divided by the difference of the two parts, the quotient may be 7.

(12) There are two vessels containing 20 gallons each; in one there are 14 gallons of pure spirit mixed with 6 gallons of water, and in the other there are 8 gallons of pure spirit mixed with 12 gallons of water: how much must be taken from each vessel, so that 20 gallons, half spirit and half water, may be obtained?

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30. *Simple equations with two unknown quantities.*

In some of the foregoing questions, *two* numbers are to be found, and in one of these questions (question 7), *four* numbers; you are supposed, however, to have arrived at the correct results without employing more than one algebraic symbol, representing one unknown quantity or number. It is often more convenient to employ two symbols (as x and y), and sometimes to use three, or even more. Generally speaking, the more you are restricted as to the number of your symbols for unknown quantities, the more you must exercise thought and ingenuity in the solution of questions in which several things are to be determined.

When two things are to be found from the question, and you represent each by a distinct symbol, the conditions of the question will always be found to be such as to furnish two distinct equations; when three things are required, then these being denoted by distinct symbols, the question will, in like manner, supply three distinct equations, and so on. It is necessary, therefore, that you should know how to solve such *simultaneous equations*, as they are called; they go by this name because they hold simultaneously, being at the same time perfectly compatible with one another and perfectly independent of one another, each expressing a distinct condition, that is, a condition not implied in the other conditions.

For solving a pair of simple equations involving two unknown quantities, there are three methods; they are as follows:

RULE I. *By equating two expressions for the same unknown quantity.*

Find an expression for one unknown quantity from the first equation, by the rule for solving an equation with only one unknown. Find an expression for *the same* unknown quantity from the second equation.

Equate these two expressions for the same thing, and you will thus have a simple equation involving only one unknown quantity, the value of which you can find by the former rule. This value substituted in one or other of the expressions for the *other* unknown, completes the solution.

II. *By the method of substitution.* Find an expression for one of the unknown quantities from either equation.

Substitute this expression for it, in place of that unknown in the *other* equation; the result will be a simple equation with only one unknown; the value of this, when determined by the former rule, is to be substituted in the expression at first obtained, when the value of the other unknown will be exhibited.

III. *By equalizing coefficients.* Multiply the two equations by such factors—the smaller the better—as will cause the coefficients of one of the unknowns to be the same in both the resulting equations, or, at least, that will prevent their differing except in signs.*

If the coefficients thus made equal have like signs, subtract one of the new equations from the other; if the equal coefficients have unlike signs, add the equations together. The result will evidently be an equation from which the unknown quantity, whose coefficients were equalized, is absent; this resulting equation may, therefore, be solved by the former rule; and one unknown being thus determined, the other may be found either by substitution or by applying the above operation to the coefficients of that other.

You perceive that whichever of these three methods be employed, the first object is to get rid of one of the unknown quantities, that is, to *eliminate* it, as it is called, in order that the former rule, for a single unknown, may be brought into operation.

Both the two examples given below for illustration are solved by all three of the methods; you will readily be able to judge for yourself which of them is best adapted for any given pair of simultaneous equations.

(1) Given the equations $\left.\begin{array}{r}3x+2y=42\\2x+3y=48\end{array}\right\}$ to find the values of x and y.

1*st method.* From the first equation,

$$x=\frac{42-2y}{3}\ldots\ldots(1)$$

From the second equation, $x=\dfrac{48-3y}{2}\ldots\ldots(2)$

Equating these two values of x,

$$\frac{42-2y}{3}=\frac{48-3y}{2}$$

Clearing fractions, $84-4y=144-9y$

Transposing, $9y-4y=144-84$

that is, $5y=60 \therefore y=12$

Substituting the value of y, now found, in the equation (1), we have,

$$x=\frac{42-24}{3}=\frac{18}{3}=6$$

Consequently, the values of the unknown quantities are $x=6$, and $y=12$.

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* The coefficients in two like terms may always be made equal by multiplying each by the other; but smaller multipliers will often bring about the equality, and, for simplicity's sake, should always be preferred when they suggest themselves.

2nd method. From the first equation, as before, $x=\frac{42-2y}{3}$. Sul stituting this expression for x in the second of the proposed equation we have

$$\frac{84-4y}{3}+3y=48$$

Clearing the fraction, $84-4y+9y=144$

Transposing, $5y=144-84=60 \therefore y=12$

Substituting this value for y, in the expression (1), we have,

$$x=\frac{42-24}{3}=6$$

3rd method. Multiply the first equation by 2, and the second by 3, i order to equalize the coefficients of x; then we have the equations,

$$6x+4y=84\ldots\ldots(1)$$
$$6x+9y=144\ldots\ldots(2)$$

Subtracting (1) from (2) $5y=60 \therefore y=12$

Substituting this value in one of the given equations, say the firs we have,

$$3x+24=42 \therefore x=\frac{42-24}{3}=6$$

Or, instead of substituting, let the first of the given equations t multiplied by 3, and the second by 2, in order to equalize the coeffi cients of y; then we have,

$$9x+6y=126$$
$$4x+6y=96$$

Subtracting lower from upper, $5x=30 \therefore x=6$

(2) Given the equations $\frac{3x-y}{4}=3$, and $\frac{x+3y}{2}=7$, to find x and y.

3rd method. Removing the fractions, the equations become

$$3x-y=12$$
$$x+3y=14$$

or, multiplying the second by 3, they are

$$3x-y=12\ldots\ldots(1)$$
$$3x+9y=42\ldots\ldots(2)$$

Subtracting (1) from (2) $10y=30 \therefore y=3$

And since from the second equation, $x=14-3y$, we have, by subst tution, $x=14-9=5$.

2nd method. From the first of the two given equations we have,

$$y=3x-12\ldots\ldots(1)$$

and substituting this value for y in the second,

$$x+9x-36=14$$

Transposing, $10x=50 \therefore x=5$

Substituting this in (1), $y=15-12=3$

1st method. From the given equations,

$$x=\frac{12+y}{3}, \text{ and also } x=14-3y\ldots\ldots(1)$$

Equating these, $\frac{12+y}{3}=14-3y$

Clearing the fraction, $12+y=42-9y$

Transposing, $10y=30 \quad \therefore \quad y=3$

Putting this value for y in (1), we have,

$$x=14-9=5$$

Of the three methods now explained, the *third* is that which will, in general, be found of the easiest application. You will not understand that it is invariably the best; a glance at the proposed equations, after you have prepared them for solution by removing fractions, &c., will always suggest which of the three is likely to involve the least amount of work. The reason why the third method is, in most cases, the preferable one, is, that it never introduces fractions, as the others usually do; and, as before remarked, fractions are more troublesome to manage than integral expressions. We shall give one more example of the application of this third method.

(3) Given $\frac{3x+4y}{5}+\frac{x}{4}=10$, and $\frac{6x-2y}{3}+\frac{y}{6}=14$, to find x and y.

The two equations, cleared of fractions, are

$$\left.\begin{array}{l} 12x+16y+5x=200 \\ 12x-\ \ 4y+\ \ y=\ \ 84 \end{array}\right\} \text{ or } \left\{\begin{array}{l} 17x+16y=200\ldots\ldots(1) \\ 12x-\ \ 3y=\ \ 84\ldots.(2) \end{array}\right.$$

As the coefficients of y are smaller numbers than those of x, we shall commence by eliminating y, that is, we shall multiply (1) by 3, and (2) by 16; we shall thus have,

$$51x+48y=\ \ 600$$
$$192x-48y=1344$$

Adding, $243x\ \ =1944 \quad \therefore \quad x=\frac{1944}{243}=8$

And from (2), $y=\frac{12x-84}{3}=4x-28=32-28=4$

Examples for Exercise.

(1) $\begin{cases} 4x+3y=31 \\ 3x+2y=22 \end{cases}$ (2) $\begin{cases} 5x+4y=55 \\ 3x+2y=31 \end{cases}$ (3) $\begin{cases} 5x-4y=19 \\ 4x+2y=36 \end{cases}$

(4) $\begin{cases} 3x+7y=\ \ 79 \\ \frac{1}{2}x-2y=-9 \end{cases}$ (5) $\begin{cases} \frac{x+8}{4}+6y=21 \\ \frac{y+6}{3}+5x=23 \end{cases}$ (6) $\begin{cases} \frac{x+1}{y}=\frac{1}{2} \\ \frac{x}{y+1}=\frac{1}{3} \end{cases}$

(7) $\begin{cases} x\ +y\ =a \\ x^2-y^2=b \end{cases}$ (8) $\begin{cases} \frac{1}{x}+\frac{1}{y}=5 \\ \frac{5}{x}-\frac{3}{y}=1 \end{cases}$ (9) $\begin{cases} 9x+6y=384 \\ 6x+9y=351 \end{cases}$

(10) $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{1}{15} \\ \frac{1}{x}+\frac{6}{y}=\frac{1}{6} \end{cases}$ (11) $\begin{cases} \frac{4}{x}+\frac{5}{y}=\frac{17}{63} \\ \frac{19}{x}+\frac{6}{y}=1 \end{cases}$ (12) $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$

(13) $\begin{cases} ax\ =by \\ x+y=c \end{cases}$ (14) $\begin{cases} (x+3)\,(y-1)=xy \\ (x-2)\,(y+1)=xy \end{cases}$

31. *Miscellaneous questions to be solved by equations involving either one or two unknown quantities.*

(1) Solve question 2, page 47, using two unknown quantities.

Suppose there is $\frac{1}{x}$ gallon of that at 20*s.*, and $\frac{1}{y}$ gallon of that at 12*s.*, the worth of the former is $\frac{20}{x}$ shillings, and the worth of the latter $\frac{12}{y}$ shillings; but the worth of the whole gallon is to be 14*s.*; hence the equations,

$$\frac{1}{x}+\frac{1}{y}=1$$
$$\frac{20}{x}+\frac{12}{y}=14$$

Multiplying the first by 12, and then subtracting the result from the second, we have,

$$\frac{8}{x}=2 \quad \therefore \quad 8=2x \quad \therefore \quad x=4$$

Again: multiplying the first by 20, and then subtracting the second from the result, we have,

$$\frac{8}{y}=6 \quad \therefore \quad 8=6y \quad \therefore \quad y=\frac{4}{3}$$

so that $\frac{1}{x}=\frac{1}{4}$, and $\frac{1}{y}=\frac{3}{4}$, the parts required.

It is plain that the first part ($\frac{1}{4}$) being determined, the second part ($\frac{3}{4}$) is at once suggested, without any work.

(2) Solve question 9, page 47, with two unknown quantities.

Suppose he takes x gallons at 9*s.* 6*d.*, and y gallons at 13*s.* 6*d.*; then the worth of the former is $19x$ sixpences, and that of the latter $27y$ sixpences; hence we have the two equations,

$$x+y=104$$
$$19x+27y=2240$$

Subtracting 19 times the first from the second, there results,

$$8y=264 \quad \therefore \quad y=33$$

and since $x=104-y, \quad \therefore \quad x=104-33=71$

consequently, there must be 71 gallons at 9*s.* 6*d.*, and 33 at 13*s.* 6*d.*

[NOTE. In the solutions to the following questions, one or two unknowns are to be employed either at the option of the learner or as the case may seem to require; but, in general, the solution will demand less thought and less work when there are as many unknowns as there are quantities to be determined.]

(1) A and B have certain sums of money; if B were to give A ten pounds, A would then have twice as much as B would have; but if A were to give the same sum to B, B would have three times as much as A: how much had each at first?

(2) Divide 24 into two such parts, that the quotient of the greater divided by the less, may be four times the quotient of the less divided by the greater

(3) There is a certain fraction such that if 3 be added to the numerator the value will be $\frac{1}{3}$, but if 1 be subtracted from the denominator the value will be $\frac{1}{5}$: what is the fraction?

(4) A person desirous of relieving some poor people by giving 2*s.* 6*d.* to each, finds that he has not money enough by 3*s.*; he therefore gives 2*s.* to each, and then has 4*s.* left: how much money had he, and how many persons did he relieve?

(5) There is a certain number, consisting of two digits, whose sum is 5, if 9 be added to the number the digits will be reversed: what is the number?

(6) A farmer has 28 bushels of barley at 2*s.* 4*d.* per bushel, with which he would mix such quantities of rye at 3*s.* per bushel, and wheat at 4*s.* per bushel, as would make 100 bushels worth 3*s.* 4*d.* per bushel: how many bushels of rye and how many of wheat must he mix with the barley?

(7) A sum of money was divided equally among a certain number of persons; if there had been three more, each would have received 1*s.* less; if there had been two fewer, each would have received 1*s.* more: how much money was divided, and among how many persons?

(8) A vessel containing 120 gallons is filled in 10 minutes by two spouts running successively; the one delivers 14 gallons a minute, and the other 9 gallons: how many minutes did each spout run?

(9) A person has £27. 6*s.* in guineas and crown-pieces, out of which he pays a debt of £14. 17*s.*, and finds that he has as many guineas left as he has paid away crown-pieces, and as many crown-pieces left as he has paid away guineas: how many of each had he at first?

(10) A labourer is hired for 30 days at 4*s.* a day for every day he works, but on condition that he forfeit 1*s.* 6*d.* for every day he is idle; at the end of the time he has to receive £3. 16*s.*: how many days did he work?

(11) Solve question (5), at page 47, with two unknown quantities.

(12) Solve questions (11) and (12), page 48, with two unknown quantities.

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32. *Simple equations with three unknown quantities.*

The methods already explained for solving a pair of simultaneous equations with two unknown quantities sufficiently suggest the mode of proceeding with three simultaneous equations involving the same number of unknowns. For taking any two of the three equations, those methods enable us to eliminate one of the unknowns, and then, combining one of the same two equations with the third, we can, in like manner, eliminate *the same* unknown from *them.*

We are thus furnished with two equations containing only two unknowns, and these may be treated as in the preceding article. No formal rule is necessary; the following specimens of a solution will amply suffice for guidance in all cases.

(1) Given $\left\{\begin{array}{l} 2x+3y+4z=29 \\ 3x+2y+5z=32 \\ 4x+3y+2z=25 \end{array}\right\}$ to find the values of x, y, and z.

First eliminate y from the first and third equations, which give

$$3y=29-2x-4z$$
$$\text{and } 3y=25-4x-2z$$
$$\therefore\ 29-2x-4z=25-4x-2z\ldots.(\text{A})$$

Next eliminate the same quantity y from the first and second e tions, by multiplying the former by 2 and the latter by 3, and tracting the results; thus,

$$\begin{array}{r}4x+6y+8z=58\\ 9x+6y+15z=96\\ \hline\end{array}$$

Difference, $5x \quad +\ 7z=38\ldots.(1)$

By equation (A) $2x \quad -\ 2z=-4$, or $x-z=-2\ldots.(2)$

From (2) $x=z-2$. Substituting this in (1) we have

$$5z-10+7z=38, \text{ or } 12z=48\ \therefore\ z=4$$

$\therefore\ x=z-2=4-2=2$; and substituting these values in either o given equations—the first, for instance—we have

$$4+3y+16=29\ \therefore\ y=\frac{29-20}{3}=3$$

Consequently the sought values are $x=2,\ y=3,\ z=4$.

(2) There are three numbers such that the difference of their d ences is 5, the sum of the numbers is 20, and their product required the numbers.

Let x, y, z, represent the numbers, then

$x-y$ and $y-z$, represent their differences,

and the difference of these differences is $x-2y+z$.

Hence by the question we have the three equations

$$x-2y+z=5\ldots.(1)$$
$$x+y+z=20\ldots.(2)$$
$$xyz=130\ldots.(3)$$

Subtracting (1) from (2), $3y=15\ \therefore\ y=5$; hence from (2) an

$$x+z=15\ldots.(4)$$
$$xz=26\ldots.(5)$$

The square of (4) is $x^2+2xz+z^2=225$

And 4 times (5) is $\quad 4xz \quad =104$

Their difference is $\quad x^2-2xz+z^2=121$

Now the first member of this equation is $(x-z)^2$ (see p. 26), an second is 11^2.

$$\therefore\ x-z=11$$

and (4) $\quad x+z=15$

Their sum is $\quad 2x=26$

Their difference is $\quad 2z=4$ $\Big\}\ \therefore\ x=13,\ z=2$

Hence the numbers are 13, 5, and 2.

In the latter part of this solution, there is an expedient adopted you will do well to attend to: there is nothing in the *rule* that p to its application; but in algebra rules can do little more than you general directions. Cases will often occur in which indeper ingenuity must be exerted, and pre-established principles be rea command. You remember that your attention was especially directed, at page 42, to the formation of the square of a binomial, that is, to the development of $(x+z)^2$ and $(x-z)^2$; the former being $x^2+2xz+z^2$, and the latter $x^2-2xz+z^2$; so that, knowing $(x+z)^2$, we may always get $(x-z)^2$ by subtracting $4xz$ from it. In the equations above, both $(x+z)^2$ and $4xz$ are given, so that $(x-z)^2$ becomes known by subtraction, and thence $x-z$ itself; and the sum $(x+z)$ and the difference $(x-z)$ of the two unknowns x, z, being thus given, the unknowns themselves are determined by addition and subtraction, as shown above.

Examples for Exercise.

(1) $\begin{cases} x+y+z=18 \\ x+3y+2z=38 \\ x+\frac{1}{3}y+\frac{1}{2}z=10 \end{cases}$ (2) $\begin{cases} 2x+3y+4z=16 \\ 3x+2y-5z=8 \\ 5x-6y+3z=6 \end{cases}$

(3) $\begin{cases} 5x-4y+2z=48 \\ 3x+3y-4z=24 \\ 2x-5y+3z=19 \end{cases}$ (4) $\begin{cases} 3x+4z=57 \\ 5x+3y=65 \\ 2y-z=11 \end{cases}$

(5) Divide £100 among A, B, and C, so that A may have £20 more than B, and B £10 more than C. [Three unknowns to be used.]

(6) A cask which held 60 gallons was filled with a mixture of brandy, wine, and cider; the cider was 6 gallons more than the brandy, and the wine was as much as the cider and one-fifth of the brandy: how many gallons were there of each?

(7) Divide 48 into four parts, such that the first increased by 3, the second diminished by 3, the third multiplied by 3, and the fourth divided by 3, may give the same results.

(8) Find three numbers, such that one-half of the first, one-third of the second, and one-fourth of the third shall together make 46; that one-third of the first, one-fourth of the second, and one-fifth of the third shall together make 35; and that one-fourth of the first, one-fifth of the second, and one-sixth of the third shall together make $28\frac{1}{3}$.

(9) A and B together can finish a piece of work in 8 days, A and C together in 9 days, and B and C together in 10 days: in how many days can each alone finish the work?

[NOTE. The principles applied in the foregoing article to the solution of simultaneous equations with three unknown quantities may be extended to n equations with n unknown quantities, since the unknowns may be eliminated one after another, as in the preceding solutions. But when the equations are numerous, the operation becomes long and troublesome; but inquiries in which more unknown quantities than three are required are of very rare occurrence in algebra.

It must be noticed that for simultaneous equations to admit of definite values for the unknowns, each equation must be distinct from and independent of all the others; if one of the equations be no more than a necessary consequence of any of the others, or be fairly derivable from those others by allowable operations, then that one cannot be regarded as a distinct equation, or as embodying any distinct conditions; so that, in reality, the number of independent equations will be fewer than the number of unknowns to be determined from them, and, therefore, the requisite number of eliminations, to lead to the definite determination of any one of the unknowns, cannot be effected—as is pretty obvious from contemplating the process. In such a case, the given conditions are insufficient.

It might happen, on the other hand, that the due number of conditions, and, therefore, of equations, might be given, and yet the conditions might be such as to involve contradictions, so that their simultaneous existence would be impossible. It might not be easy to discover this want of congruity in a set of simultaneous equations; but we need never attempt such discovery; we may proceed with our eliminations on the assumption that our equations are perfectly independent and perfectly compatible; if this assumption be correct, we shall discover the hidden values of our unknown symbols; if it be incorrect, our results will either give no definite values, or will else give such as will be palpably absurd and contradictory. The two examples following will serve as practical illustrations of what is here said.

Insufficient Conditions. *Incompatible Conditions.*

$$(10)\ \begin{cases} 3x-2y+\ 5z=14 \\ 2x+y-\ 8z=10 \\ 6x-4y+10z=28 \end{cases} \qquad (11)\ \begin{cases} 3x-2y+5z=14 \\ 2x+y-8z=10 \\ 8x-3y+2z=35 \end{cases}]$$

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33. *Quadratic equations involving only one unknown quantity.*

The equations considered in the preceding articles are all *simple* equations; they are not called *simple* because they are so *easy*, but because each unknown enters in its simplest or lowest power, or at least, whenever any higher power, as the square, the cube, &c., occurred in the equation, it was always removable by subtraction, or division, or some other of the operations common to simple equations of the ordinary form.

A quadratic equation is one which always involves the second power, or square, of the unknown quantity, for the removal of which, and the consequent reduction of the quadratic to a simple equation, a special and peculiar operation is necessary, requiring a distinct rule, derivable from the following considerations:

1. The square of a binomial, such as $x+a$, or $x-a$, always consists of *three* terms, thus:—

$$(x+a)^2=x^2+2ax+a^2\ldots.(1)$$
$$(x-a)^2=x^2-2ax+a^2\ldots.(2)$$

2. If only the first and second of these three terms be written down, we can always complete the square by supplying the wanting third term; for we see that this third term is nothing but *half the coefficient of x*, in the second term, *squared;* thus the coefficient of x in (1) is $2a$, the coefficient of x in (2) is $-2a$; the half of the former is a, and the half of the latter $-a$; and, in either case, the square of this half is a^2, the square of a quantity being always *plus*, whether the sign of the quantity itself be + or −.

Having therefore the first term x^2, and the second term, before us, we can always complete the square to which these two terms belong. For example, suppose the two terms x^2+6x are given to complete the square. Here, as the coefficient of x is 6, of which the half is 3, we know that the wanting third term is 9; the complete square being

$$x^2+6x+9=(x+3)^2$$

the *root* of the square thus completed being simply x, together with the half coefficient used in completing it. If the incomplete square had been x^2-6x, then the wanting term $(-3)^2$ would have been equally 9; thus,

$$x^2-6x+9=(x-3)^2$$

the root of the square being $x-3$.

The following are a few examples of this.

Complete the square, and write down the root of it, in the cases here given:—

(1) x^2+8x, (2) x^2-12x, (3) x^2-2x, (4) x^2+10x,
(5) x^2-5x, (6) x^2+3x, (7) x^2+x, (8) x^2-x.

The roots of these squares, when completed, are

(1) $x+4$, (2) $x-6$, (3) $x-1$, (4) $x+5$, (5) $x-\frac{5}{2}$,
(6) $x+\frac{3}{2}$, (7) $x+\frac{1}{2}$, (8) $x-\frac{1}{2}$.

---

34. *To solve a quadratic equation.*

RULE 1. Place all the unknown terms on one side of the equation, and all the known on the other, as in simple equations; fractions and radicals being cleared as before.

2. If the first term, or that containing x^2, have a coefficient other than unity, remove it by division.

3. The leading term being now simply x^2, take half the coefficient of x in the second term, and add the square of this half-coefficient to both sides; the unknown side will then be a complete square.

4. Extract the root of this square, which is merely x with the half-coefficient united to it by its own sign; extract, in like manner, the root of the known side; we shall then have a simple equation, from which x may be easily determined.

NOTE. Remember that the square root of a number takes the double sign ± (see page 36).

(1) Required the values of x in the equation $x^2-6x=7$.

As the coefficient of x^2 is unit, and the known and unknown quantities properly arranged, no preparatory operation is requisite before applying precept 3 of the rule; therefore, completing the square, by adding $(-3)^2$ to each side,

$$x^2-6x+9=16$$

Extracting the root, $x-3=\pm4 \quad \therefore \quad x=3\pm4$

Consequently the value of x is indifferently either $x=7$ or $x=-1$. If either value be put for x in the proposed equation, its conditions are satisfied; thus, putting 7 for x, the equation is $49-42=7$; and putting -1 for x, it is $1+6=7$.

(2) Find the values of x in the equation $2x^2+8x-30=60$.

Transposing, $2x^2+8x=90$

Dividing by 2, $x^2+4x=45$

Completing the square, $x^2+4x+4=49$

Extracting the root, $x+2=\pm 7$

$$\therefore\quad x=-2\pm 7=5 \text{ or } -9.$$

(3) Given the equation $3x^2-3x+9=\frac{25}{3}$ to find x.

Transposing, $3x^2-3x=-\frac{2}{3}$

Dividing by 3, $x^2-x=-\frac{2}{9}$

Completing the square, $x^2-x+\frac{1}{4}=-\frac{2}{9}+\frac{1}{4}=\frac{1}{36}$

Extracting the root, $x-\frac{1}{2}=\pm\frac{1}{6}$

$$\therefore\quad x=\frac{1}{2}\pm\frac{1}{6}=\frac{2}{3} \text{ or } \frac{1}{3}$$

(4) Given the equation $\frac{x}{3}-\frac{\sqrt{x}}{2}=\frac{3}{2}$ to find the value of x.

Multiplying by 6, to clear fractions,

$$2x-3\sqrt{x}=9$$

From this we may remove the radical in the usual way; but it is better to put y for $\sqrt{x}$, and thus change the equation into

$$2y^2-3y=9$$

$$\therefore\quad y^2-\frac{3}{2}y=\frac{9}{2}.$$

Completing the square, $y^2-\frac{3}{2}y+\frac{9}{16}=\frac{9}{2}+\frac{9}{16}=\frac{81}{16}$

Extracting the root, $y-\frac{3}{4}=\pm\frac{9}{4}$

$$\therefore\quad y=\frac{3}{4}\pm\frac{9}{4}=3 \text{ or } -\frac{3}{2}$$

$$\therefore\quad x=y^2=9 \text{ or } \frac{9}{4}$$

NOTE. Every equation of the form $x^{2n}+ax^n=b$, may be solved as a quadratic by putting y for x^n, and therefore y^2 for x^{2n}, as in this example, where $n=\frac{1}{2}$.

(5) Given the equation $\frac{2\sqrt{x}+2}{4+\sqrt{x}}=\frac{4-\sqrt{x}}{\sqrt{x}}$, to find x.

Clearing fractions, and remembering that the sum of two quantities multiplied by their difference is the difference of the squares of those quantities, we have,

$$2x+2\sqrt{x}=16-x$$

Transposing, $3x+2\sqrt{x}=16$

Or putting y for $\sqrt{x}$, $3y^2+2y=16$

Dividing by 3, $y^2+\frac{2}{3}y=\frac{16}{3}$

Completing the square, $y^2+\frac{2}{3}y+\frac{1}{9}=\frac{16}{3}+\frac{1}{9}=\frac{49}{9}$

Extracting the root, $y+\frac{1}{3}=\pm\frac{7}{3}$

$$\therefore\quad y=-\frac{1}{3}\pm\frac{7}{3}=2 \text{ or } -\frac{8}{3}$$

$$\therefore\quad x=y^2=4 \text{ or } \frac{64}{9}$$

(6) Given the equation $x^6+4x^3=12$, to find x.

Putting, agreeably to the preceding note, $x^3=y$, we have

$$y^2+4y=12$$

Completing the square, $y^2+4y+4=16$

Extracting the root, $y+2=\pm4$

$$\therefore\quad y=-2\pm4=2 \text{ or } -6$$

$$\therefore\quad x=\sqrt[3]{y}=\sqrt[3]{2} \text{ or } \sqrt[3]{-6}$$

(7) Given $ax^2+bx=c$, to find expressions for x.

Dividing by a, $x^2+\frac{b}{a}x=\frac{c}{a}$

Completing the square, $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{c}{a}+\frac{b^2}{4a^2}$

Extracting the root, $x+\frac{b}{2a}=\sqrt{\frac{4ac+b^2}{4a^2}}=\frac{\sqrt{(4ac+b^2)}}{2a}$

$$\therefore\quad x=\frac{-b\pm\sqrt{(4ac+b^2)}}{2a}$$

[NOTE. This is a general *formula* for the solution of every quadratic equation, whatever may be the particular values of a, b, c. The values of x are always two-fold; because, when the operation implied in $\sqrt{}$ is performed, the result is ±. If the coefficients a, b, c* are so related that $4ac+b^2=0$, then the irrational part of the expression vanishes, and the two *roots*, as the values of x are called, are *equal*, namely, each is $-\frac{b}{2a}$; when the relation is such that $4ac+b^2$ is negative, then the two roots are *imaginary*. In all other cases, the roots are *real* and *unequal*. It is clear that a being positive, the roots can never be either equal or imaginary, unless c is negative, since whatever be the sign of b, b^2 is necessarily positive.

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* There is no impropriety in calling c a *coefficient*, it is the coefficient of x^0 in the equation $ax^2+bx=cx^0$, which is the same as $ax^2+bx=c$, because $x^0=1$ (p. 34).

If the general equation above were written thus, $ax^2+bx+c=0$, which, indeed, is in a form more convenient for the discussion of the *theory* of quadratic equations, then the above general solution would, of course, be,

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

and we should have the following conditions, namely,

If $b^2=4ac$, the two roots are real and equal.
$b^2 \, 7 \, 4ac$,* „ „ real and unequal.
$b^2 \angle 4ac$, „ „ imaginary.

If one root be imaginary, it is obvious that the other must be imaginary also, because the irrational part of the expression for x enters equally into both values.]

Examples for Exercise.

[The learner may refer to the foot-note to Ex. 11 for some useful hints.]

(1) $2x^2=4x+6$ (2) $x^2-8x+15=0$ (3) $\frac{(x+6)(x+7)}{2}=15x$

(4) $x^2-x=1$ (5) $3x^2-102=x$ (6) $(x+2)^2=6x+2$

(7) $\frac{x-1}{x-2}+\frac{x-2}{x-1}=1$ (8) $\frac{x-4}{\sqrt{x}+2}=x-8$ (9) $\sqrt{(x+12)}=\frac{12}{\sqrt{(x+5)}}$

(10) $\sqrt{(x+21)}+\sqrt{(x+21)}=12$ (11) $x-1=2+\frac{2}{\sqrt{x}}$†

(12) $\frac{1}{x^3}+\frac{1}{\sqrt{x^3}}=2$ (13) $\frac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}}=x$ (14) $\frac{x}{a}+\frac{a}{x}=\frac{2}{a}$

(15) $(2x+6)^{\frac{1}{2}}+(2x+6)^{\frac{1}{4}}=6$ (16) $\frac{3(x-5)}{4}-\frac{x-1}{x+2}=2x-17$

(17) $5x^{\frac{2}{3}}+7x^{\frac{1}{3}}=108$ (18) $\frac{3x+5}{3x-5}-\frac{3x-5}{3x+5}=\frac{135}{176}$

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* The signs 7 and ∠ stand for the words *greater than* and *less than*, respectively.

† If $\frac{1}{x}$ be added to each side of the equation $x-2=1+\frac{2}{\sqrt{x}}$, the first side will become the square of $\sqrt{x}-\frac{1}{\sqrt{x}}$, and the second the square of $1+\frac{1}{\sqrt{x}}$. In Ex. 12, y may be put for $\sqrt{x^3}$, or for $\frac{1}{\sqrt{x^3}}$. In Ex. 15, y may be put for $(2x+6)^{\frac{1}{4}}$; for $\sqrt[4]{(x+21)}$ in Ex. 10; and for $x^{\frac{1}{3}}$ in Ex. 17. In dealing with Ex. 8, remember that if the sum of two quantities is a divisor, and the difference of their squares a dividend, the quotient will be the difference of the quantities themselves. To Ex. 13, the principle explained at p. 45 may be applied.

35. *Second method of solving a quadratic equation.*

The rule given at page 57 for the solution of a quadratic equation is that generally taught in the books, and applied in actual practice; but, as in order to complete the square, the coefficient of x^2 must be made unit, and then the half-coefficient of x squared, it is plain that fractions are likely to be introduced at an early stage of the work, and to encumber our expressions to the end.

The method now to be explained never *introduces* fractions, and is, therefore, to be preferred in all cases such as those just adverted to, and which, indeed, are those of by far the most frequent occurrence in practice; we may pave the way to it thus:—

Taking the general form $ax^2+bx=c$, let each side be multiplied by $4a$, it then becomes $4a^2x^2+4abx=4ac$, which it will be convenient to write thus:—

$$(2ax)^2+2b(2ax)=4ac$$

Written in this way, you perceive that the first member is in a form well adapted for the completion of the square without fractions; the unknown quantity being regarded as $2ax$. Proceeding by the former rule, the complete square is,

$$(2ax)^2+2b(2ax)+b^2=4ac+b^2$$

And therefore the root, $\quad 2ax+b=\sqrt{(4ac+b^2)} \ldots . (1)$

You thus see that any quadratic $ax^2+bx=c$ is reduced to the simple equation (1), without the introduction of fractions, at a single step, by following these precepts, namely:—

RULE II. 1. Take twice the coefficient of x^2 in the proposed quadratic; this will be the coefficient of x in the reduced simple equation, the first term of which is thus found at once.

2. To this first term unite, with its own sign, the coefficient of x in the proposed equation; the first side of the derived simple equation will thus be obtained.

3. The second side is found thus: multiply the second side of the proposed (namely c) by 4 times the coefficient of x^2 (that is, by $4a$), *add* the square of the next coefficient (that is, of b) to the result, and cover the two with the radical sign.

This rule merely describes in words how,

From the quadratic equation, $\quad ax^2+bx=c$

There results the simple equation, $2ax+b=\sqrt{(4ac+b^2)}$

And this is the only step that need be committed to memory; the convenience of it will be sufficiently seen by employing it in some of the equations solved by the former rule.

Ex. 3, page 58, $3x^2-3x+9=\frac{25}{3}$, or $9x^2-9x+27=25$

$$\therefore \quad 9x^2-9x=-2$$

$\therefore$ Rule II. $\quad 18x-9=\sqrt{(-72+81)}=\pm 3$

$$\therefore \quad x=\frac{9\pm 3}{18}=\frac{2}{3} \text{ or } \frac{1}{3}$$

Ex. 4, page 58, after removing radical, $2y^2-3y=9$

$\therefore$ Rule II. $\quad 4y-3=\sqrt{(72+9)}=\pm 9$

$$\therefore \quad y=\frac{3\pm 9}{4}=3 \text{ or } -\frac{3}{2}$$

Ex. 5, after removing radical, $3y^2+2y=16$

$\therefore$ Rule II. $\quad 6y+2=\sqrt{(192+4)}=\pm 14$

$$\therefore \quad y=\frac{-2\pm 14}{6}=2 \text{ or } -\frac{8}{3}$$

The method of solution here illustrated is that which will always be employed in the subsequent parts of the present work, except in the comparatively few instances in which fractions are not introduced by the common method of completing the square. As an exercise, let the rule be applied to the equations in examples 4, 5, and 17, at page 60, putting, in the last example, y for $x^{\frac{1}{3}}$.

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36. *Questions producing quadratic equations with one unknown quantity.*

(1) A and B set off from the same place, and at the same time, to travel a distance of 150 miles. By travelling 3 miles an hour faster, A arrives at his journey's end 8 hours and 20 minutes before B: at what rate per hour did A and B travel?

Let $x=$ miles per hour B travels

then $x+3=$ " " A "

and $\frac{150}{x}=$ hours occupied by B

$\frac{150}{x+3}=$ " " A

Now, by the question, if $8\frac{1}{3}$ hours be added to A's time, the sum will be equal to B's; hence we have the equation,

$$\frac{150}{x+3}+8\tfrac{1}{3}=\frac{150}{x}$$

that is, $$\frac{150}{x+3}+\frac{25}{3}=\frac{150}{x}$$

or dividing by 25, $$\frac{6}{x+3}+\frac{1}{3}=\frac{6}{x}$$

Clearing fractions, $18x+x^2+3x=18x+54$

that is, $x^2+3x=54$

$\therefore$ Rule II. $\quad 2x+3=\sqrt{(216+9)}=\sqrt{225}=\pm 15$

$$\therefore \quad x=\frac{-3\pm 15}{2}=6 \text{ or } -9$$

The *negative* value of x is, of course, excluded by the nature of the question; $\therefore$ $x+3=9$; consequently, A travels 9 miles per hour, and B 6.

You will observe that although the solution of a quadratic equation always furnishes *two* values for the unknown quantity, yet that the question in which the quadratic originates may be so restricted as to render only one of these values admissible; the *equation* being free from the restrictions of the *question* as to admissible values.

(2) What two numbers are these of which the difference is 2 and the difference of the cubes 98?

Let the two numbers be x and $x+2$, then, by the question,

$$(x+2)^3-x^3=98$$

that is, $$6x^2+12x+8=98$$

Transposing, $$6x^2+12x=90$$

Dividing by 6, $$x^2+2x=15$$

Completing the square, $x^2+2x+1=16$, $\therefore\ x+1=\pm 4$, $\therefore\ x=3$ or -5, $\therefore\ x+2=5$, or -3; hence the numbers are either 3 and 5, or -3 and -5.

(3) Divide the number 20 into two parts such that their product may be 105.

Let x be one part, then $20-x$ is the other, and, by the question,

$$20x-x^2=105$$

or, $$x^2-20x=-105$$

Completing the square, $$x^2-20+100=-5$$

Extracting the root, $$x-10=\sqrt{-5}$$

$$\therefore\ x=10\pm\sqrt{-5}$$

As the value of x contains an imaginary quantity, we conclude that the solution of the question is impossible in real numbers; it is impossible to divide 20 into two parts of which the product is 105. This is one use of imaginary quantities; their entrance into the answer showing that the question implies an impossibility.

Examples for Exercise.

(1) Divide 33 into two parts such that their product may be 162.

(2) A person bought a certain number of sheep for £120, and finds that if he had had 8 more for the same money, the price of each would have been 10*s.* less: how many did he buy?

(3) What two numbers are those of which the sum is 6, and the sum of the cubes 72?

(4) Is it possible to divide 126 into two parts such that their product may be 4000?

(5) A and B start at the same time and from the same place to travel a distance of 300 miles; A travels 1 mile an hour faster than B, and arrives at the journey's end 10 hours before him: at what rate per hour did each person travel?

(6) A person bought cloth for £33. 15*s.*, and sold it at £2. 8*s.* per piece, thus gaining by the transaction as much as one piece cost him: how many pieces were there?

(7) A company at a tavern had £8. 15*s.* to pay; but before the bill was settled, two of them left, in consequence of which those who remained had each 10*s.* more to pay: how many persons were there at first?

(8) There are two numbers whose product is 120; if 2 be added to the less, and 3 subtracted from the greater, the product will still be 120: required the numbers.

(9) A gamester won the first game as much money as he had in his pocket, at the second game he won *5s.* more than the square root of what he then had, at the third he won the square of all he then had, and found his money to amount to £112. *16s.*: how much had he at first?

(10) There are two numbers whose difference is 4, and which are such that if one be taken for numerator and the other for denominator of a fraction, and then the former be taken for denominator and the latter for numerator, the sum of the square roots of the two fractions is also 4: required the numbers.

37.—SIMULTANEOUS EQUATIONS WITH TWO UNKNOWN QUANTITIES.

1. *When one is a simple equation and the other a quadratic.*

RULE I. From the simple equation find an expression for one of the unknowns in terms of the other and the known quantities.

Substitute this value in the quadratic, from which one unknown will thus be eliminated, and the equation reduced to a quadratic with a single unknown.

(1) $\left.\begin{aligned} 3x-4y&=3 \\ x^2-y^2&=16 \end{aligned}\right\}$ From the first equation $x=\frac{3+4y}{3}$; and putting this value for x in the second, we have

$$\left(\frac{3+4y}{3}\right)^2-y^2=16\text{; that is }\frac{9+24y+16y^2}{9}-y^2=16$$

Multiplying by 9, and transposing,

$$7y^2+24y=135$$

$$\therefore \text{ RULE II.} \quad 14y+24=\sqrt{(135\times 28+24^2)}=\sqrt{4356}=\pm 66$$

$$\therefore y=\frac{-24\pm 66}{14}=3,\text{ or }-6\tfrac{3}{7}$$

$$\therefore x=\frac{3+4y}{3}=5,\text{ or }-7\tfrac{4}{7}$$

hence the values of x and y are either

$$\left.\begin{aligned} x&=5 \\ y&=3 \end{aligned}\right\} \text{ or } \left\{\begin{aligned} x&=-7\tfrac{4}{7} \\ y&=-6\tfrac{3}{7} \end{aligned}\right.$$

(2) $\left.\begin{aligned} 2x-3y&=1 \\ 2x^2+xy-5y^2&=20 \end{aligned}\right\}$ From the first equation $x=\frac{1+3y}{2}$. Substituting this for x in the second, we have

$$2\left(\frac{1+3y}{2}\right)^2+\frac{1+3y}{2}y-5y^2=20$$

$$\text{or, }\frac{1+6y+9y^2}{2}+\frac{y+3y^2}{2}-5y^2=20$$

Multiplying by 2, and transposing, $2y^2+7y=39$

$\therefore$ RULE II. $4y+7=\sqrt{(312+49)}=\sqrt{361}=\pm19$

$$\therefore\ y=\frac{-7\pm19}{4}=3, \text{ or } -6\tfrac{1}{2}$$

$$\therefore\ x=\frac{1+3y}{2}=5, \text{ or } -9\tfrac{1}{4}$$

hence the values of x and y are either

$$\left.\begin{matrix}x=5\\y=3\end{matrix}\right\} \text{ or } \left\{\begin{matrix}x=-9\tfrac{1}{4}\\y=-6\tfrac{1}{2}\end{matrix}\right.$$

NOTE.—As observed at page 54, by the exercise of a little address, a simpler and neater solution may often be obtained, than would be furnished by implicitly following the directions of a formal *rule*. In quadratics with two unknowns, instances very frequently occur where rules are of but little avail, and where the solution can be effected only by the independent skill and ingenuity of the algebraist. You will see the advantage of departing from rule in the following example.

(3) $\left.\begin{matrix}x^2y^2+4xy=96\\x+y\quad=6\end{matrix}\right\}$ The first equation is $(xy)^2+4(xy)=96$ $\therefore$ completing the square,

$$(xy)^2+4\,(xy)+4=100$$

Extracting the root, $xy+2=\pm10$ $\therefore\ xy=8, \text{ or } -12$

Squaring the second equation,

$$x^2+2xy+y^2=36$$

Subtracting $4xy\quad=32 \text{ or } -48$

we have $x^2-2xy+y^2=\ 4 \text{ or } 84$

Extracting the root, $x-y=\pm2 \text{ or } \sqrt{84}=2\sqrt{21}$

$\therefore$ adding and subtracting $x+y=6$

we have $2x=8, \text{ or } 4, \text{ or } 6\pm2\sqrt{21}$

$2y=4, \text{ or } 8, \text{ or } 6\mp2\sqrt{21}$

hence the values of x and y are

$$\left.\begin{matrix}x=4\\y=2\end{matrix}\right\} \text{ or } \left.\begin{matrix}x=2\\y=4\end{matrix}\right\} \text{ or } \left\{\begin{matrix}x=3\pm2\sqrt{21}\\y=3\mp2\sqrt{21}\end{matrix}\right.$$

It thus appears that x and y have each *four* values. If the rule had been implicitly followed, the first equation, after substituting in it the expression $6-y$ for x, or $6-x$ for y, would evidently have been one of the *fourth* degree; and every equation has as many *roots* as there are units in the number marking its degree; that is, a simple equation has one root, a quadratic two, a cubic three, and so on. But the proof of this general proposition belongs to the "Theory of Equations."*

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* See the "Analysis and Solution of Cubic and Biquadratic Equations," by the author of this work.

2. *When both are quadratic equations.*

38. If both equations are of the second degree, the principles of algebra, which suffice for the solution of every quadratic with one unknown quantity, will often be found to be inadequate, whatever artifices or expedients be employed to bring the proposed simultaneous equations within the range of those principles. When the elimination of one of the unknowns is effected, the resulting equation, involving the other unknown, will frequently rise to the third or fourth degree, requiring special and more advanced methods for its solution. There are, however, two classes of simultaneous quadratics which are manageable by the rules already established, namely, *homogeneous equations*, and *symmetrical equations.* These may be treated as in the following examples.

Homogeneous quadratics. A pair of equations are said to be homogeneous when each unknown term in both is of the same dimensions, that is, when each term consists of the same number of unknown factors; thus, the terms x^3, $3x^2y$, $2xy^2$, $4y^3$ are all of *three* dimensions, because each contains three unknown factors. In homogeneous quadratics, each term is of *two* dimensions.

RULE 1. Put for one of the unknown quantities an unknown multiple of the other, that is, for y, put zx, or for x, zy, then the square of this other will enter each of the unknown terms, so that an expression for it may be deduced from each equation.

2. Equate these two expressions; you will then have a quadratic equation in which the factor z is the only unknown quantity, and from which z, and thence x and y, may be determined.

(1) $\left.\begin{array}{r}2x^2+3xy+y^2=20\\ 5x^2+4y^2=41\end{array}\right\}$ Put zx for y, then the equations become

$$2x^2+3zx^2+z^2x^2=20 \quad \therefore\ x^2=\frac{20}{2+3z+z^2}$$

$$5x^2+4z^2x^2=41 \quad \therefore\ x^2=\frac{41}{5+4z^2}$$

Equating these two expressions for x^2, we have

$$\frac{20}{2+3z+z^2}=\frac{41}{5+4z^2}$$

Clearing fractions, $\quad 100+80z^2=82+123z+41z^2$

Transposing, $\quad 39z^2-123z=-18$

or, $\quad 13z^2-41z=-6$

$\therefore$ Rule II. $\quad 26z-41=\sqrt{(-312+41^2)}=\pm 37$

$$\therefore\ z=\frac{41\pm 37}{26}=3 \text{ or } \frac{2}{13}$$

$$\therefore\ x^2=\frac{41}{5+4z^2}=\frac{41}{41}=1, \text{ or } \frac{41\times 13^2}{5\times 13^2+4^2}=8\tfrac{1}{21}$$

$$\therefore\ x=\pm 1, \text{ or } \sqrt{8\tfrac{1}{21}}$$

$$\therefore\ y=zx=\pm 3, \text{ or } \pm\frac{2}{13}\sqrt{8\tfrac{1}{21}}$$

Examples for Exercise.

(1) $\left.\begin{array}{r} x^2+xy=77 \\ xy-y^2=12 \end{array}\right\}$ (2) $\left.\begin{array}{r} 4x^2-2xy=12 \\ 3xy+2y^2=\ \ 8 \end{array}\right\}$ (3) $\left.\begin{array}{r} x^2+xy-12=0 \\ xy-2y^2-1=0 \end{array}\right\}$

(4) $\left.\begin{array}{l} x^2+4y^2=256-4xy \\ 3y^2-x^2=39 \end{array}\right\}$ (5) $\left.\begin{array}{r} 6x^2-5xy+2y^2=12 \\ 3x^2+2xy-3y^2=-3 \end{array}\right\}$

(6) $\left.\begin{array}{r} 2x^2-3xy+y^2=4 \\ 2xy-3y^2=x^2-9 \end{array}\right\}$

39. *Symmetrical quadratics.* An equation is said to be symmetrical when the unknowns entering it may be interchanged without altering the equation; for instance, $x+y=a$ is a symmetrical equation, and so is $x^2-axy+y^2=b$, because, if x be changed into y, and y into x, the equation remains unaltered. The general method of solving a pair of symmetrical quadratics, and which may often be successfully applied to symmetrical equations of higher degrees, is as follows:—

Rule. Substitute $u+v$ for x, and $u-v$ for y, in each equation; you will thus get a pair of equations containing the new unknowns u and v, instead of x and y, and from which the values of u and v may be found, as in the examples below.

Note. It has been more than once observed that though general rules may always be followed with safety, yet that, in particular cases, certain expedients may often be employed with advantage; these must be suggested by practice, and a careful observation of the peculiarities of the example before us; they cannot possibly be embodied in rules. The example following will illustrate this remark.

(1) $\left.\begin{array}{r} x^2+y^2-x-y=18 \\ xy+x+y=19 \end{array}\right\}$ Substituting $u+v$ for x and $u-v$ for y, agreeably to the rule, the equations become

$$\left.\begin{array}{r} (u+v)^2+(u-v)^2-2u=18 \\ (u+v)(u-v)+2u=19 \end{array}\right\} \text{ or } \left\{\begin{array}{l} u^2+v^2-\ u=9 \\ u^2-v^2+2u=19 \end{array}\right.$$

Adding these, to eliminate v, $2u^2+u=28$

$$\therefore \text{ Rule II.} \qquad 4u+1=\sqrt{225}=\pm 15 \;\therefore\; u=\frac{7}{2} \text{ or } -4$$

$$\therefore\; v^2=9+u-u^2=\frac{1}{4}, \text{ or } -11 \quad \therefore\; v=\pm\frac{1}{2}, \text{ or } \sqrt{-11}$$

$$\therefore\; x=u+v=\frac{7}{2}\pm\frac{1}{2}, \text{ or } -4\pm\sqrt{-11}\text{; that is, } x=4, \text{ or } 3, \text{ or } -4\pm\sqrt{-11}$$

$$y=u-v=\frac{7}{2}\mp\frac{1}{2}, \text{ or } -4\mp\sqrt{-11}\text{; that is, } y=3, \text{ or } 4, \text{ or } -4\mp\sqrt{-11}$$

On account of the symmetrical form of each equation, if $x=a$, $y=b$, be one pair of values, then $x=b$, $y=a$, must be another pair.

Let us now try to solve the preceding equations independently of the general rule.

By adding them together, after doubling the second, we have

$$x^2+2xy+y^2+x+y=56$$

that is, $\qquad (x+y)^2+(x+y)=56$

$$\therefore \text{ Rule II.} \qquad 2(x+y)+1=\sqrt{225}=\pm 15$$

$$\therefore\; x+y=\frac{-1\pm 15}{2}=7 \text{ or } -8\ldots(1)$$

Hence by substitution, the proposed equations become

$$x^2+y^2=25, \text{ or } 10$$
$$xy=12, \text{ or } 27$$

Subtracting twice the second from the first,

$$x^2-2xy+y^2=1, \text{ or } -44$$
$$\therefore\ x-y=\pm 1, \text{ or } 2\sqrt{-11}$$

But (1), $x+y=7, \text{ or } -8$

$\therefore$ adding and subtracting, $2x=7\pm 1, \text{ or } -8\pm 2\sqrt{-11}$

$$2y=7\mp 1, \text{ or } -8\mp 2\sqrt{-11}$$

$$\therefore\ x=4,\ 3, \text{ or } -4\pm\sqrt{-11};\ y=3,\ 4, \text{ or } -4\mp\sqrt{-11}$$

Examples for Exercise.

(1) $\left.\begin{array}{r}\dfrac{x^2}{y}+\dfrac{y^2}{x}=18\\ x+y=12\end{array}\right\}$

(2) $\left.\begin{array}{r}x^3+y^3=4\\ x+y=2\end{array}\right\}$

(3) $\left.\begin{array}{r}x^3+y^3=189\\ x^2y+xy^2=180\end{array}\right\}$

(4) $\left.\begin{array}{r}x^2+y^2+x+y=18\\ xy=\ \ 6\end{array}\right\}$

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40. Ratios, Proportions, and Progressions.

When one quantity is divided by another of the same kind, the quotient is always an abstract number; this quotient is also called the *ratio* of the two quantities, the dividend is called the *antecedent* of the ratio, and the divisor the *consequent*, so that dividend, divisor, and quotient, in common division, answer to antecedent, consequent, and ratio, here; thus the ratio of 8 to 4 is 2, and is the same as the ratio of 4 to 2, or that of 2 to 1. In like manner, the ratio of a to b may be expressed by the fraction $\frac{a}{b}$, or by $a \div b$, but it is more usual to express it thus, $a : b$, which differs from $a \div b$ only by the absence of the little mark between the dots. The antecedent and consequent (in this case a and b) are called the *terms* of the ratio.

Proportion is an equality of ratios; it consists, therefore, of *four* terms; thus, if $\frac{a}{b}=\frac{c}{d}$, the four terms which enter these equal ratios, namely, a, b, c, d, are said to be in proportion; the equality of the ratios is usually expressed thus, $a : b :: c : d$, and sometimes, though less frequently, thus, $a : b = c : d$, and it is expressed in words by saying *a is to b, as c is to d*. The first and last of the four terms of a proportion are called the *extremes*, and the two intermediate terms the *means*.

As a proportion is only another form of writing and expressing two equal fractions, and two equal fractions only another form of expression for a proportion, the properties of equal fractions become the properties of a proportion; and *vice versâ*. Of these properties, the following are of the most frequent application.

41. *Theorems respecting Proportion.*

Theorem 1. If four numbers are in proportion, the product of the extremes is equal to the product of the means, and, conversely, if the product of one pair of numbers be equal to the product of another, the factors of one product will be the extremes, and those of the other product the means of a proportion.

1. Let $a : b :: c : d$; then $\frac{a}{b}=\frac{c}{d}$ $\therefore$ $ad=bc$

2. Let $ad=bc$ $\therefore$ $ad+db=bc+db$ $\therefore$ $\frac{a}{b}=\frac{c}{d}$; that is,

$$a : b :: c : d$$

If the proportion be $a : b :: b : c$, the three quantities a, b, c are said to be in continued proportion, and from what is just proved the product of the extremes of three such quantities is equal to the square of the mean, that is, $ac=b^2$. Moreover, because $\frac{a}{b}=\frac{b}{c}$, it follows, multiplying by $\frac{b}{c}$, that $\frac{a}{c}=\frac{b^2}{c^2}$, $\therefore$ $\frac{a}{c}=\frac{a^2}{b^2}$, that is, the first is to the third as the square of the first to the square of the second.

Theorem 2. If four quantities are in proportion, they are also in proportion when taken *inversely;* that is, the second is to the first as the fourth to the third.

Let $a : b :: c : d$, that is, $\frac{a}{b}=\frac{c}{d}$; then, since the reciprocals of two equal quantities must be equal, $\frac{b}{a}=\frac{d}{c}$ $\therefore$ $b : a :: d : c$.

Theorem 3. If four quantities of the same kind are in proportion, they are also in proportion when taken *alternately;* that is, the first is to the third as the second to the fourth.

Let $a : b :: c : d$, that is, $\frac{a}{b}=\frac{c}{d}$. Multiply by $\frac{b}{c}$, then $\frac{a}{c}=\frac{b}{d}$; that is, $a : c :: b : d$.

It was necessary to stipulate that the quantities be all of the same kind, for if any two differed in kind, those two could have no ratio. From the definitions of ratio and proportion, the first and second terms of a proportion must be of one kind, and the third and fourth also of one kind, but the two pairs of terms need not be of the *same* kind unless alternation is to be applied to them; the ratios $\frac{a}{b}$, $\frac{a}{c}$ employed above imply that a, b, c are like in kind, and the ratio $\frac{b}{d}$ implies that b and d are also like in kind; and thus all four of the terms must be like in kind.

Theorem 4. If $a : b :: c : d$, then also

$$a\pm mb : a\pm nb :: c\pm md : c\pm nd$$

For since $\frac{a}{b}=\frac{c}{d}$ $\therefore$ $\frac{a}{b}\pm m=\frac{c}{d}\pm m\ldots.(1)$

and, $\frac{a}{b}\pm n=\frac{c}{d}\pm n\ldots.(2)$

Dividing (1) by (2),

$$\frac{a\pm mb}{a\pm nb}=\frac{c\pm md}{c\pm nd}\ \therefore\ a\pm mb : a\pm nb :: c\pm md : c\pm nd$$

where m and n are any values whatever.

[This is only a particular case of a much more general theorem, namely:—

If two fractions are equal, as $\frac{a}{b}=\frac{c}{d}$, we may replace the terms a, b, of the first, by any two expressions which are such that each term of both is of the same dimensions, as respects a and b; provided only we replace the terms c, d, of the second, by two expressions involving c and d in a similar manner; thus, from

$$\frac{a}{b}=\frac{c}{d}, \text{ we may infer that } \frac{3a^2-5ab+2b^2}{4a^2+2ab-b^2}=\frac{3c^2-5cd+2d^2}{4c^2+2cd-d^2}$$

For let $c=ma$ (m being some number whole or fractional), then $d=mb$, otherwise the fractions could not be equal. Putting these values for c and d in the second of the changed fractions, it becomes $\frac{3m^2a^2-5m^2ab+2m^2b^2}{4m^2a^2+2m^2ab-m^2b^2}$, which is the same as the first.]

Theorem 5. If any number of quantities of the same kind are proportionals, then as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let there be $a : b :: c : d :: e : f$, &c.

Put $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ &c. $=m$; then $a=mb$, $c=md$, $e=mf$, &c.

$$\therefore\ a+c+e+\text{ \&c. }=m(b+d+f+\text{ \&c.})$$

$$\therefore\ \frac{a+c+e+\text{ \&c.}}{b+d+f+\text{ \&c.}}=m=\frac{a}{b}=\frac{c}{d}, \text{ \&c.}$$

$$\therefore\ a : b :: a+c+e+\text{ \&c. } : b+d+f+\text{ \&c.}$$

Besides these, many other theorems may be easily deduced; the following are obvious consequences of the equality $\frac{a}{b}=\frac{c}{d}$.

6. If the first and second (a, b) of four proportionals be multiplied or divided by any quantity (m), and the third and fourth (c, d) by any quantity (n), the results are proportionals.

7. If the first and third, and then the second and fourth, be in like manner multiplied or divided, the results are proportionals; that is $\frac{ma}{nb}=\frac{mc}{nb}$, whatever be m and n.

8. Like powers or roots of four proportionals are also proportionals; that is $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, whatever be n.

9. If several sets of four are proportionals, then, if the corresponding antecedents and consequents be multiplied together, the products will be proportionals; for if

$$\left.\begin{array}{c} \frac{a}{b}=\frac{c}{d} \\ \frac{e}{f}=\frac{g}{h} \\ \frac{i}{k}=\frac{l}{m} \\ \&c.\ \&c. \end{array}\right\} \begin{array}{l} \text{then } \frac{aei\ \&c.}{bfk\ \&c.}=\frac{cgl\ \&c.}{dhm\ \&c.} \\ \therefore aei\ \&c. : bfk\ \&c. :: cgl\ \&c. : dhm\ \&c. \end{array}$$

NOTE 1. Theorem 1 establishes the truth of the rule known in arithmetic as the Rule of Three, in which three terms of a proportion are given to find the fourth; for since the product of the first and fourth is equal to that of the second and third, it follows that, to find the fourth, we have only to divide the product of the second and third by the first.

NOTE 2. The proportion discussed in this article is usually called *geometrical proportion.* If four quantities be such that the first exceeds or falls short of the second by the same quantity that the third exceeds or falls short of the fourth, the four are said to be in *arithmetical proportion;* so that, as the former kind of proportion implies equal *quotients,* the latter implies equal *differences;* thus a, $a+d$, b, $b+d$, are in arithmetical proportion; the sum of the extremes is evidently equal to the sum of the means. Whenever the word *proportion* is used, without any prefix, *geometrical* proportion is always to be understood.

[*Variation* is a term frequently used, instead of *proportion,* in such cases as these, namely:—the price of a commodity varies with its weight; the amount of work varies with the time expended on it, and so on; the meaning being, that one price is to another as the weight at the former price to the weight at the latter; or, that one amount of work is to another as the time expended on the former to that expended on the latter. In all such cases, *two* quantities suffice to indicate the proportion instead of *four,* the mark $\propto$, which stands for *varies as,* being placed between them; thus, $A \propto B$ implies that A varies as B. If A, B be two corresponding states of these mutually dependent quantities, and A', B' any two other corresponding states, then the operation $A \propto B$ is equivalent to the statement $A : A' :: B : B'$.

Here A is said to vary *directly* as B; but if $A \propto \frac{1}{B}$, that is, if $A : A' :: \frac{1}{B} : \frac{1}{B'}$, then A is said to vary *inversely* as B; in which case, by multiplying the third and fourth terms by BB', the proportion may be written $A : A' :: B' : B$.

If $A \propto \frac{B}{C}$, that is, if $A : A' :: \frac{B}{C} : \frac{B'}{C'}$, then A is said to vary directly as B and inversely as C. And if $A \propto BC$, then A is said to vary *conjointly* as B, C.

What is here said may be practically illustrated by reference to a triangle, thus,

1. If the altitude be invariable, Area $\propto$ Base

2. „ area „ „ Base $\propto \dfrac{1}{\text{Perp.}}$

3. If neither be invariable, Base $\propto \dfrac{\text{Area}}{\text{Perp.}}$

4. „ „ Area $\propto$ Base × Perp.

From what is said above, it appears that a case of variation may always be converted into one of proportion; but in solving problems, it is, in general, sufficient to convert the variation into an equation, whenever the varying quantities are like in kind. When we say that one such quantity varies as another, we only affirm that, throughout all their mutual changes, their *ratio* remains the same; so that, calling this constant ratio m, the expression $x \propto y$ may always be replaced by $x = my$. We shall give two examples.

(1) If $z \propto y$, and $y \propto x$, prove that $x \propto \sqrt{yz}$. Since z varies as y, and y as x, it follows that z varies as x; so that both z and y vary as x.

$$\text{Put } \left.\begin{aligned} y &= mx \\ z &= nx \end{aligned}\right\} \therefore\ yz = mnx^2 \ \therefore\ x = \frac{1}{\sqrt{mn}}\sqrt{yz} \ \therefore\ x \propto \sqrt{yz}$$

(2) The conditions of variation being as above, prove that

$$x + y + z \propto \sqrt{yz} + \sqrt{xz} + \sqrt{xy}.$$

$$\text{Here } x + y + z = (1 + m + n)x \ \therefore\ x = \frac{x + y + z}{1 + n + m} \ldots\ldots (1)$$

$$\text{Because} \quad \left.\begin{aligned} x &= x \\ y &= mx \\ z &= nx \end{aligned}\right\} \quad \therefore \quad \begin{aligned} \sqrt{xy} &= \sqrt{m}\cdot x \\ \sqrt{xz} &= \sqrt{n}\cdot x \\ \sqrt{yz} &= \sqrt{mn}\cdot x \end{aligned}$$

$$\therefore\ x = \frac{\sqrt{yz} + \sqrt{xz} + \sqrt{xy}}{\sqrt{mn} + \sqrt{n} + \sqrt{m}} = \frac{x + y + z}{1 + n + m}, \text{ by (1) above}$$

$$\therefore\ x + y + z \propto \sqrt{yz} + \sqrt{xz} + \sqrt{xy}]$$

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42. ARITHMETICAL PROGRESSION.

An arithmetical progression is a series of quantities of which each differs from that immediately following by the same constant difference.

If a represent the first term, d the common difference (which may be either positive or negative), and n the number of terms, then the general form of the arithmetical progression is

$$a,\ a + d,\ a + 2d,\ a + 3d,\ \ldots . a + (n - 1)d$$

To investigate a formula for the sum (S) *of an arithmetical series.*

Putting l for the last term, that is, putting

$$l = a + (n - 1)d \ldots . \text{(I.)}$$

$$S = a + (a + d) + (a + 2d) + (a + 3d) + \ldots\ldots + l$$

or, writing the n terms in reverse order,

$$S = l + (l - d) + (l - 2d) + (l - 3d) + \ldots\ldots + a$$

Adding these two equivalent expressions together,

$$2S=(a+l)+(a+l)+(a+l)+(a+l)+\ldots\ldots(a+l)=n(a+l)$$

$$\therefore\ S=\frac{n}{2}(a+l)=\frac{n}{2}\{2a+(n-1)d\}\ \text{by (I.)}\ldots\ldots\text{(II.)}$$

Any three of the quantities, a, d, l, n, being given, the fourth may be found by formula (I.); and any four of the quantities a, d, l, n, S being given, the fifth may be found by formula (II.). The rule for S may be thus expressed.

Rule. Multiply the sum of the first and last terms by half the number of terms; the product will be the sum of the series. The last term is the first increased by as many times the difference as there are terms following the first.

(1) The first term of a decreasing arithmetical series is 199, the common difference 3, and the number of terms 67: required the sum of the series.

Here $a=199$, $d=-3$, $n=67$ $\therefore$ $l=199-66\cdot3=1$.

$$\therefore\ S=\frac{67}{2}\cdot200=6700$$

(2) Find the sum of one hundred terms of the series 1, 2, 3, 4, 5, &c.

$$a=1,\ d=1,\ n=100,\ l=100,\ \therefore\ S=50\cdot101=5050.$$

(3) The fifth term of an arithmetical progression is 13, and the eleventh term is 31: required the series.

By formula (I.),

$$\begin{array}{l} a+4d=13\ldots\ldots(1) \\ a+10d=31\ldots\ldots(2) \\ \hline \therefore\ \ 6d=18\quad\therefore\ \ d=3 \end{array}$$

$\therefore$ (1), $a+12=13$ $\therefore$ $a=1$, and $\therefore$ the series is

1, 4, 7, 10, 13, &c.

This problem may be solved a little differently, by regarding 13 as the first term and 31 as the seventh term of the series, and thence finding d from formula (I.), thus,

$$31=13+6d,\ \therefore\ d=\frac{18}{6}=3$$

so that by adding 3 continually to the 13 up to 31, we get the part of the series to the right of the term 13, and subtracting 3, we get the four terms to the left. The five terms thus interposed between 13 and 31 are called arithmetical means, as in the following example.

(4) It is required to insert four *arithmetical means* between 7 and 13, that is, 7 being the first term and 13 the sixth, to supply the intermediate terms.

As $a=7$, $n=6$, and $l=13$, we have, by formula (I.),

$$13=7+5d,\ \therefore\ d=\frac{6}{5}=1\tfrac{1}{5}$$

Hence the four means are $8\frac{1}{5}$, $9\frac{2}{5}$, $10\frac{3}{5}$, $11\frac{4}{5}$.

Examples for Exercise.

(1) Required the sum of ten terms of the series 1, 5, 9, 13, &c.

(2) Required the sum of seven terms of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, &c.

(3) The fourth term is 9, and the seventh 15: required the arithmetical progression.

(4) Insert two arithmetical means between 3 and 9.

(5) Insert three arithmetical means between 6 and 26.

(6) How many terms of the series 1, 5, 9, &c., must be added together to make 190?

(7) Find the sum of an arithmetical series of which the first term is 10, the common difference $\frac{1}{3}$, and the number of terms twenty-one.

(8) A hundred stones being placed on the ground in a straight line, at the distance of two yards apart, how far must a person travel who brings them one by one to a basket placed two yards behind the first stone?

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43. GEOMETRICAL PROGRESSION.

A series of quantities are said to be in geometrical progression when the quotient of any one and that which immediately precedes it is uniformly the same.

If a represent the first term of such a series, r the constant quotient or ratio, and n the number of terms, the entire progression will be,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5, \ldots .\, ar^{n-1}$$

From inspecting this general form, the following properties become obvious, namely,

1. The product of any two terms is equal to the product of any other two equidistant from them, thus,

$$ar^2 \times ar^4 = ar \times ar^5 = a \times ar^6$$

And if the number of terms be odd, the product of the extremes is equal to the square of the middle term, or the *mean;* thus $a \times ar^4 = (ar^2)^2$.

2. The last (or nth term) is equal to the product of the first term, and that power of the ratio which expresses the number of terms *minus* 1, that is, calling the last term l,

$$l = ar^{n-1} \ldots\ldots \text{(I.)}$$

To investigate an expression for the sum (S) *of a geometrical series.*

$$S = a + ar + ar^2 + ar^3 + \ldots\ldots\ldots ar^{n-1}$$
$$rS = \quad ar + ar^2 + ar^3 + ar^4 + \ldots .\, ar^{n-1} + ar^n$$

Subtracting, $(r-1)S = ar^n - a = rl - a$ (by I.) $\therefore\ S = a\dfrac{r^n - 1}{r-1} = \dfrac{rl - a}{r-1}$..(II.)

This latter formula for S, expressed in words, is the rule following, namely:—

RULE. Multiply the last term by the common ratio; from the product subtract the first term, and divide the remainder by the ratio minus 1. The last term is the first multiplied as often by the ratio as there are terms following the first.

If r be a whole number greater than unity, or an improper fraction, the terms of the series continually increase, and at length become infinitely great; if r be a proper fraction, they continually diminish, and at length become infinitely small; an infinitely small quantity cannot sensibly differ from 0, since whatever value we assign to it short of 0 must be too great. Hence the infinite decreasing series may be represented thus,

$$a+ar+ar^2+\ldots\ldots+0$$

where 0 may be regarded as the last term; for including the 0 cannot add to the *sum*, even should it be disputed that a zero-term *can* be rigorously arrived at. Hence, putting 0 for l in the formula for S, we have the following expression for the sum of an *infinite* series, changing S into Σ, to distinguish an infinite from a finite series.

$$\Sigma=\frac{a}{1-r}\ldots\ldots\text{(III.)}$$

This formula evidently involves less computation than the formula (II.), so that we may affirm that it is more easy to compute the sum of a decreasing geometrical series, when continued to infinity, than to compute only a few of its leading terms;—a statement that, in the absence of sufficient evidence, would seem to be paradoxical. The rule for summing an infinite decreasing series is as follows:—

RULE. Divide the first term by 1 minus the ratio.

(1) Required the sum of five terms of the series,

$$1-4+16-64+\ \&c.$$

Here $a=1,\ n=5,\ r=-4\ \therefore\ l=ar^{n-1}=(-4)^4=256$

$$\therefore\ S=\frac{rl-a}{r-1}=\frac{-1025}{-5}=205$$

(2) Required the sum of eight terms of a geometrical series of which the first term is $\frac{1}{3}$, and the common ratio $\frac{1}{2}$. Find also the sum of the same series to infinity.

$$a=\frac{1}{3},\ r=\frac{1}{2},\ n=8,\ \therefore\ l=ar^{n-1}=\frac{1}{3}\left(\frac{1}{2}\right)^7=\frac{1}{384}$$

$$\therefore\ S=\frac{rl-a}{r-1}=\left(\frac{1}{768}-\frac{1}{3}\right)\div-\frac{1}{2}=\frac{255}{384}$$

Also, $$\Sigma=\frac{a}{1-r}=\frac{1}{3}\times\frac{2}{1}=\frac{2}{3}$$

so that the sum of the series continued to infinity is just double of the first term. And it is plain, from the formula, that the sum of any geometrical series to infinity is always equal to the first term multiplied by the reciprocal of what the ratio wants of unity; thus, when the ratio is $\frac{2}{3}$, the sum Σ is $3a$, when the ratio is $\frac{3}{5}$, the sum is $\frac{5}{2}a$, when it is

$-\frac{1}{3}$, the sum is $\frac{3}{4}a$, and so on. The sum is always the same in sign as the first term, whether the ratio is a positive fraction or a negative one.

Examples for Exercise.

(1) Find the sum of twelve terms of the series 1, 2, 4, &c.

(2) Find the sum of twelve terms of the series 1, 3, 9, &c.

(3) It is required to insert three geometrical means between 2 and 32; that is, to complete the series of which 2 is the first term and 32 the fifth.

(4) Insert five geometrical means between 2 and $\frac{1}{32}$.

(5) Required the sum of the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$ &c., to infinity.

(6) Required the fraction equal to the recurring decimal 666...., that is, sum the series $\frac{6}{10}+\frac{6}{100}+\frac{6}{1000}+$ &c.

NOTE. The fraction equal to any number, part of which is a recurring or circulating decimal, may be found as follows:

Let the number be 347·584584....

$$\text{Put } F = 347{\cdot}584\ldots.$$
$$\therefore 1000\,F = 347584{\cdot}584\ldots.$$
$$\therefore 999\,F = 347237 \quad \therefore F = \frac{347237}{999}$$

Here F is multiplied by 1000 in order to bring a complete period before the decimal point.

Again, let
$$F = 4{\cdot}7543543\ldots.$$
$$10\,F = 47{\cdot}543\ldots.$$
$$10000\,F = 47543{\cdot}543\ldots.$$
$$\therefore 9990\,F = 47496 \quad \therefore F = \frac{47496}{9990} = \frac{7916}{1665}$$

This process suggests the rule given in books on Arithmetic.

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44. *Miscellaneous questions involving proportion and progression.*

(1) The sum of 500*l.* is to be divided between two persons, so that their shares may be to each other as 7 to 8; required the share of each.

Let the shares be x and y; then by the question,

$$x+y=500 \quad \therefore y=500-x$$
$$\text{and} \quad x:y::7:8$$
$$\therefore (\textit{Theo. } 1\text{, p. } 69.) \quad 8x=7y \quad \therefore 8x=3500-7x$$
$$\therefore 15x=3500 \quad \therefore x=\frac{3500}{15}=233\tfrac{1}{3}$$
$$\therefore \quad y=500-x=266\tfrac{2}{3}$$

Hence the shares are 233*l.* 6*s.* 8*d.* and 266*l.* 13*s.* 4*d.*

(2) Find three numbers in geometrical progression such that their sum shall be 7 and the sum of their squares 21.

Let the numbers be x, y, and z; then by the question

$$x+y+z=7 \quad \ldots .(1)$$

$$xz=y^2 \quad \ldots .(2) \quad (\textit{Theo. } 1, \text{ p. } 69.)$$

$$x^2+y^2+z^2=21 \ldots .(3)$$

From (1) $x+z=7-y$ $\therefore$ $x^2+2xz+z^2=49-14y+y^2$

Subtracting (2) $x^2+xz+z^2=49-14y$

that is, from (2) $x^2+y^2+z^2=49-14y=21$, by (3)

$\therefore$ $14y=49-21=28$ $\therefore$ $y=2$

$\therefore$ (1) and (2) $x+z=5$ $\therefore$ $x^2+2xz+z^2=25$

$xz=4$ $\therefore$ $4xz \quad =16$

$\therefore$ $x^2-2xz+z^2= \ 9$

$\therefore$ $x-z=\pm 3$, and as $x+z=5$ $\therefore$ $x=\frac{5\pm 3}{2}$, $z=\frac{5\mp 3}{2}$; that is,

$x=4$, or 1; $y=2$; $z=1$, or 4: so that the numbers are 1, 2, 4.

(3) Find four numbers in arithmetical progression such that the product of the extremes may be 22 and that of the means 40.

Let x be the less extreme, and y the common difference, then the four numbers are x, $x+y$, $x+2y$, and $x+3y$; and by the question,

$$x^2+3xy=22 \ldots .(1)$$

$$x^2+3xy+2y^2=40 \ldots .(2)$$

Subtracting (1) from (2), $2y^2=18$ $\therefore$ $y^2=9$ $\therefore$ $y=\pm 3$

$\therefore$ (1) $x^2\pm 9x=22$ $\therefore$ $2x\pm 9=\sqrt{(88+81)}=\pm 13$

$\therefore$ $x=\frac{\mp 9\pm 13}{2}=2$ or -2; or 11 or -11

Hence the numbers are 2, 5, 8, 11, or -2, -5, -8, -11.

The other two values of x give the same numbers in reverse order. It must be observed that the values $x=2$ and $x=-11$ arise from taking the positive value of y; and $x=-2$ and $x=11$ arise from taking the negative value of y; and in uniting the values of x and y in the expressions $x+y$, $x+2y$, $x+3y$, care must be taken not to combine them so as to violate these dependent conditions. You must always take note of the connection of one unknown with another dependent on it as to the *signs*, and not pair them together in opposition to that connection.

Examples for Exercise.

(1) What number is that to which if 3, 5, and 8 be severally added, the three sums shall be in geometrical progression?

(2) How many terms of the series 2, 5, 8, 11, &c., must be taken in order that the sum may amount to 26?

(3) If a person were to begin the year by giving away a farthing the first day, a halfpenny the second, three farthings the third, and so on: how much money will he thus have disposed of at the end of the year?

(4) Find three numbers in geometrical progression such that their product may be 64, and the sum of their cubes 584.

(5) Find two numbers in the ratio of 2 to 3, and such that the difference of their squares may be 245.

(6) A hare is 50 of her leaps before a greyhound in pursuit of her; she takes four leaps to the greyhound's three; but two of the greyhound's leaps make three of the hare's: how many leaps must the greyhound take to catch the hare?

(7) What two numbers are those whose difference, sum, and product are as the numbers 2, 3, and 5, respectively?

(8) A corn-factor wishes to mix wheat-flour which cost him 10*s.* a bushel with barley-flour which cost him 4*s.* a bushel, in such proportion as to gain $43\frac{3}{4}$ per cent. by selling the mixture at 11*s.* a bushel: required the proportion.

(9) There is a number consisting of three digits which are in arithmetical progression; if the number be divided by the sum of the digits, the quotient will be 26, but if 198 be added to the number, the digits will occur in reverse order. What is the number? [A number of three digits, x, y, z, is expressed thus, $100x+10y+z$.]

(10) There are three positive numbers in geometrical progression whose sum is 13, and the product of the mean and sum of the extremes is 30: required the numbers.

(11) Prove that the geometrical mean between $\sqrt{(n^2+1)}+\sqrt{(1-n^4)}$ and $\sqrt{(n^2+1)}-\sqrt{(1-n^4)}$ is n times the arithmetical mean.

(12) The following are three infinite series, namely,

$$S_1=1+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\ldots\ldots$$

$$S_2=1-\frac{1}{r}+\frac{1}{r^2}-\frac{1}{r^3}+\ldots\ldots$$

$$S_3=1+\frac{1}{r^2}+\frac{1}{r^4}+\frac{1}{r^6}+\ldots\ldots$$

Prove that $S_1\times S_2=S_3$

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45. HARMONICAL PROPORTION AND PROGRESSION.

[Besides the two kinds of proportion and progression considered in the foregoing articles, there is a third kind, called harmonical proportion and harmonical progression.

Four quantities are said to be in harmonical proportion when the first has to the fourth the same ratio as the difference between the first and second has to the difference between the third and fourth; thus, a, b, c, d are in harmonical proportion (or, as they are, for brevity, called, *harmonicals*) when

$$a:d::a-b:c-d, \text{ or } a(c-d)=d(a-b)$$

so that any three of the harmonicals being given the fourth may be found.

When, of the four harmonicals, the two means are equal, as a, b, b, c, then the three, a, b, c, are in harmonical progression, and we have

$$a:c::a-b:b-c, \text{ or } a(b-c)=c(a-b)$$

$$\therefore\ ab+bc=2ac\ \therefore\ b=\frac{2ac}{a+c}$$

so that the harmonic mean (b) between two quantities (a, c) is twice their product divided by their sum.

When the series consists of more terms than three, and is such that every consecutive three form an harmonical progression, the whole series is called an harmonical progression. Every harmonic series is closely connected with an arithmetical progression, since the reciprocals of the terms of one series are always the terms of the other,

For let a, b, c, d, &c. be an harmonic series, and call the reciprocals of the terms a', b', c', d', &c.; then, from the equation above, we have the following conditions, namely,

$$ab+bc=2ac, \quad bc+cd=2bd, \quad \&c. \ldots\ldots (1)$$

or, dividing each equation by the product of the three quantities entering it,

$$\frac{1}{c}+\frac{1}{a}=\frac{2}{b}, \quad \frac{1}{d}+\frac{1}{b}=\frac{2}{c}, \quad \&c. \ldots\ldots\ldots (2)$$

that is, $\quad c'+a'=2b', \quad d'+b'=2c', \quad \&c. \ldots\ldots\ldots (3)$

so that a', b', c', d, &c. are in arithmetical progression.

And conversely, by taking any arithmetical progression a', b', c', d', &c., we might, from the conditions (3), get (2), and thence (1); that is, we might show that the reciprocals of an arithmetical progression form an harmonical progression. It is from this connection between the two kinds of series that the term *harmonical* has been applied, for musical strings, of equal thickness and tension, produce perfect harmony, when sounded together, only when their lengths are as the reciprocals of the arithmetical progression 1, 2, 3, 4, 5, &c., that is, the lengths of the musical strings must be as 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c.; and hence the propriety of calling the reciprocals of such progressions harmonical or musical progressions.

When any two terms, however wide apart, are given, we can, as in the other kinds of progression, always discover the intermediate terms or means; for from the relation just established, we shall only have to take the reciprocals of the given extremes, to find the proposed number of *arithmetical* means between them, and then to take the reciprocals of these means. Thus, suppose we have to insert two harmonic means between 1 and 2.

Two arithmetical means between 1 and $\frac{1}{2}$, are $\frac{5}{6}$ and $\frac{2}{3}$; therefore, the two harmonic means are $\frac{6}{5}$ and $\frac{3}{2}$; the series being 1, $\frac{6}{5}$, $\frac{3}{2}$, 2.

There is no general formula for the *sum* of an harmonic series; the allied arithmetical series gives no aid here, and we should have to proceed by actual addition of the terms; but all the problems respecting harmonicals, of any particular application, can be solved by help of the corresponding arithmetical series.]

THE BINOMIAL THEOREM.

46. The binomial theorem is a general expression for the development of any power or root of a binomial. It is one of the most impor-

tant theorems in algebra, as it not only enables us to write out the development or expansion of $(a+x)^n$, whatever be n, without the trouble of actually performing the involution or extraction indicated by n, but it also makes known to us a remarkable law, to which the several terms of the result are always subject, and which law affords us great assistance in many theoretical and practical inquiries connected with the more advanced parts of algebra.

In the portions of the subject discussed in the preceding pages, the principal object in view, after the fundamental rules of operation were established, has been the determination of unknown values from certain assigned relations between them and given quantities; that is, we have been mainly occupied in solving equations, and in working questions. We now enter upon a department of algebra of a different character; instead of equations proposed for solution, we shall have to deal chiefly with *identities*,* and to investigate the changes of form under which certain algebraic expressions may be written, without any regard to particular values of the general symbols.

The changes here spoken of are not objects of mere algebraic curiosity, or of mere speculative interest; they subserve the purposes of science in the most important of its practical applications; there are certain symbolical expressions presented to the eye in a brief and compact form, which, nevertheless, indicate operations of very laborious performance. The aim is to unravel or develope these expressions, to exhibit their component parts in detail, to discover the hidden laws of their composition, and thus to facilitate their actual calculation, whenever the general symbols receive particular interpretations. Of these developments, that of the expression $(a+x)^n$—the BINOMIAL THEOREM of Sir Isaac Newton—is of primary consequence, not only on account of its intrinsic value, but because of the aid it affords in the decomposition of other algebraical forms. In the following articles, we shall give the investigation of this important theorem in as complete and simple a manner as we are able.

But we must first establish a preliminary principle of essential use in this class of inquiries.

Preliminary Theorem.

47. If the series $A+Bx+Cx^2+Dx^3+$&c., the terms being either finite in number, or infinite, be equal to the series $A'+B'x+C'x^2+D'x^3+$&c., whatever value be given to x; then, provided only that the

---

* *Identities* are, of course, *equations*, but equations are not in general identities, as they usually imply that the symbols are subject to conditions, and are thus limited to particular values, determinable from what is called the *solution* of the equation. It would be absurd to speak of the solution of an identical equation, which fixes no conditions; $(x+y)^2=x^2+2xy+y^2$; $(x+y)(x-y)=x^2-y^2$, &c., are identical equations, the second member of each being nothing more than the first member differently expressed, so that the equation holds for *all* values of x and y, without limitation.

coefficients are none of them infinite in value, the coefficients of the *like* powers of x will always be equal; that is, $A=A'$, $B=B'$, $C=C'$, &c.

For since the two series are equal for all values of x, without exception, we are at liberty to put for x the extreme value $x=0$, for which value, however, all the terms after the first in each must vanish, since the coefficients are finite.

Consequently $A=A'$. Knowing this, we may conclude that

$$Bx+Cx^2+Dx^3+\text{ &c.}=B'x+C'x^2+D'x^3+\text{ &c.}$$

or, dividing by x, that

$$B+Cx+Dx^2+\text{ &c.}=B'+C'x+D'x^2+\text{ &c.}$$

But, for the value $x=0$, these, as before, become simply $B=B'$, so that the following equality must have place, namely—

$$Cx+Dx^2+\text{ &c.}=C'x+D'x^2+\text{&c.}$$

$$\text{or } C+Dx+\text{ &c.}=C'+D'x+\text{ &c.}$$

which, for $x=0$, becomes simply $C=C'$. And by proceeding uniformly in this way, we infer the equality of the coefficients of all the like powers; that is, that

$$A=A',\ B=B',\ C=C',\ D=D',\ \text{&c.}^*$$

Investigation of the Binomial Theorem.

48. The theorem we have to prove is this, namely, that whatever be the exponent n,

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}x^3+\text{&c.}$$

If we were to divide each side of this assumed equality by a^n, the first would be $\left(\frac{a+x}{a}\right)^n$, that is, $\left(1+\frac{x}{a}\right)^n$, or putting z for $\frac{x}{a}$, it would be $(1+z)^n$; and in like manner dividing, and substituting, with respect to the second side, we should have only to prove the truth of the following more simple form, namely—

$$(1+z)^n=1+nz+\frac{n(n-1)}{2}z^2+\frac{n(n-1)(n-2)}{2\cdot 3}z^3+\text{ &c.}$$

It is plain that if we can prove this, we shall only have to put $\frac{x}{a}$ for z, and then to multiply each side by a^n in order to get the theorem as announced above. We shall prove first that no fractional or negative exponent can enter the development of $(1+z)^n$, whatever be n.

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* Some very able algebraists object to this form of proof; they say it involves a division by 0; but where there is any division by 0 in the steps above, I am at a loss to discover. In the first step, x is assumed to be 0, in virtue of the liberty, allowed by the hypothesis, to assume *any value* for x, the zero-value included; the result is, that, in the proposed pair of series, A' must be A; we therefore return to the series, and, availing ourselves of this knowledge, expunge the first term from each, and then divide by the factor x, common to both, without restricting the x by any hypothesis whatever, and so on to the end.

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48. I. *The development of* $(1+z)^n$ *is always of the form*

$$(1+z)^n = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c. \dots . (1)$$

1. Suppose first that n is a positive whole number; then, as $(1+z)^n$ is a positive power determinable by the repeated multiplication of $1+z$ by itself, it is plain that the first term of the final product must be 1, and that all the powers of z in that product must be positive integers; if neither fractional nor negative exponents occur in the *factors*, they cannot possibly, from the nature of multiplication, occur in the *product*. In the case supposed, therefore, the development is necessarily of the form (1).

When we say that the development is of the *form* (1), you are to understand that the development is affirmed to conform to (1) as a model; that if we only knew what values to put for the unknown symbols P, Q, R, &c., the form might be made *identical* to the correct development, without disturbing the exponents of z. The determination of the actual values of these symbols will be made matter of especial consideration hereafter; at present we aim only at establishing the *form* of the development.

2. Suppose, secondly, that n is a positive fraction, as $\frac{p}{q}$; then, $(1+z)^{\frac{p}{q}}$ indicates that after the power p of $1+z$ has been found, the root q of that power is to be extracted. Now the power p, as already proved, is of the form (1); and whatever root of (1) be extracted, since by repeated multiplications of that root by itself, (1) is reproduced, it follows, from the nature of multiplication, that the leading term of it must be 1, and that no fractional exponent can enter it, for whatever fractional exponent enters an expression whose first term is 1, the same must enter every power of that expression. For instance, whatever power we take of $1 + ax + cx^{\frac{7}{3}} + dx^3 +$ &c., the fractional power $x^{\frac{7}{3}}$ will always reappear in the result; and if the vacant place be filled up by a zero-term, and the expression be written thus,

$$1 + ax + 0x^2 + cx^{\frac{7}{3}} + dx^3 + \&c.$$

the fractional exponent $\frac{7}{3}$ will always occur in the same *place* in the power as here in the root, namely, in the fourth place; and the same is evidently the case whether the leading term of the root be 1 or any other number. Of course the proper place of a power is that which it occupies when all the powers in the expression are arranged in regular ascending or descending order. Hence no fractional exponent can enter the development of $(1+z)^{\frac{p}{q}}$. [The absence of negative exponents is proved for all cases at the end of the next paragraph.]

3. Suppose, lastly, that n is negative, and let it be either integral or

fractional; then the development of $(1+z)^{-n}$, or, which is the same thing, the development of $\frac{1}{(1+z)^n}$ will be the quotient of 1 by an expression of the form (1). The first term of this quotient is therefore 1; and since the first remainder cannot contain any fractional power of z, neither can the next term of the quotient, neither, therefore, can the second remainder, nor, in consequence, the next term of the quotient, and so on. Hence, generally, whether n be integral or fractional, positive or negative, the form of the development will always be

$$(1+z)^n = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.$$

provided, at least, that a negative exponent cannot possibly enter the development. If negative exponents be supposed possible, let $-k$ represent the largest of them; then, if every term containing z were to be divided by z^{-k}, no negative exponents could occur in the quotient, because each exponent in the dividend being numerically less than $-k$, when each is increased by k, all would be positive; consequently, the development would be of the form,

$$(1+z)^n = 1 + (a + bz^\alpha + cz^\beta + dz^\gamma + \ \&c.)z^{-k}$$

$\therefore$ multiplying by z^k,

$$(1+z)^n z^k = z^k + (a + bz^\alpha + cz^\beta + dz^\gamma + \ \&c.)$$

The two sides being identical, we may put any value for z; put $z=0$, then $0=a$; that is, the coefficient of the supposed negative power is 0; in other words, such negative power cannot enter the development. In all cases, therefore (1), is the form the development assumes.

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49. II. *Determination of the coefficient* (P) *in the general form* (1).

1. When the coefficients P, Q, R, &c., are known for any particular value of the exponent n, their values for the next higher exponent $n+1$, may be easily found by actual multiplication; thus, if

$$(1+z)^n = 1 + pz + qz^2 + \ \&c.$$

then multiplying by $\quad 1+z$

$$\begin{array}{r} 1 + pz + qz^2 + \ \&c. \\ z + pz^2 + \ \&c. \\ \hline \end{array}$$

we have $(1+z)^{n+1} = 1 + (p+1)z + \ \&c.$

Hence the difference $n-p$ between *any* exponent n and the coefficient p in the second term of the development, is the same as the difference between $n+1$ and the coefficient $p+1$ in the second term of the succeeding development. Now when $n=1$, the development is $1+1z$, when $n=2$, it is $1+2z+z^2$, in which cases we see that the difference between the exponent and coefficient of the second term is *nothing;* hence, for integral positive exponents, the coefficient of the second term is always equal to the exponent; that is, *when* n *is integral and positive,*

$$(1+z)^n = 1 + nz + Qz^2 + Rz^3 + Sz^4 + \text{\&c.} \dots (2)$$

Thus $(1+z)^3 = 1 + 3z + Qz^2 + Rz^3 + \text{\&c.}$, $(1+z)^4 = 1 + 4z + Qz^2 + Rz^3 + \text{\&c.}$ $(1+z)^5 = 1 + 5z + Qz^2 + Rz^3 + \text{\&c.}$; but the values of the other coefficients Q, R, S, &c. are still unknown to us.

(2) The same equality between the exponent and the coefficient of the second term also exists when the exponent is a negative integer; for, since by (2)

$$\frac{1}{(1+z)^n} = \frac{1}{1 + nz + Qz^2 + Rz^3 + \text{\&c.}}$$

we have only to actually perform the division here indicated as far as two terms in the quotient to establish this truth; thus,

$$\begin{array}{rll} 1 + nz + Qz^2 + \text{\&c.}\ ) & 1 & (\ 1 - nz + \text{\&c.} \\ & \underline{1 + nz + \text{\&c.}} & \\ & -nz - \text{\&c.} & \\ & -nz - \text{\&c.} & \end{array}$$

Hence, whether the integer n be positive or negative, (2) is still the form of the development.

(3) The same equality exists when the exponent is either a positive or negative fraction. For let $\pm\frac{p}{q}$ be put for n; then (1)

$$(1+z)^{\pm\frac{p}{q}} = 1 + Pz + Qz^2 + Rz^3 + \text{\&c.}$$

For brevity put z' for $Pz + Qz^2 + \text{\&c.}$, then

$$(1+z)^{\pm\frac{p}{q}} = 1 + z' \quad \therefore \quad (1+z)^{\pm p} = (1+z')^q$$

that is, substituting for these expressions their developments as far as they have been ascertained,

$$1 \pm pz + \text{\&c.} = 1 + qz' + \text{\&c.}$$

or, replacing z' by the expression it stands for,

$$1 \pm pz + \text{\&c.} = 1 + q(Pz + Qz^2 + \text{\&c.}) + \text{\&c.}$$

This equation is of course true whatever be the value of z, for it is an identical equation; consequently, the coefficients of like powers of z are equal (47).

$$\therefore\ \pm p = qP \quad \therefore\ \pm\frac{p}{q} = P$$

that is, the coefficient of the second term of the development is still equal to the exponent.

It is thus proved generally, that whether n be whole or fractional, positive or negative,

$$(1+z)^n = 1 + nz + Qz^2 + Rz^3 + Sz^4 + \text{\&c.} \dots (3)$$

or, replacing z by its value $\frac{x}{a}$, and then multiplying all by a^n, that

$$(a+x)^n = a^n + na^{n-1}x + Qa^{n-2}x^2 + Ra^{n-3}x^3 + Sa^{n-4}x^4 + \text{\&c.} \dots (4)$$

in which development the subsequent coefficients Q, R, S, &c., remain still to be determined.

50. III. *Determination of the coefficients* Q, R, S, *&c.*

The investigation thus far has been tolerably easy, the present concluding portion of it will require a little more algebraical address; the first step of it involves one of those artifices or expedients for which the ingenuity of the investigator is often taxed in the higher departments of algebra; and, throughout, the preliminary theorem (at 47) is turned to important account.

In the identity (3), substitute $b(1+x)$ for z, which we may, of course, do, as z is altogether arbitrary; we may put for it whatever we please at any stage of the investigation. We shall thus have

$$\{1+b(1+x)\}^n=1+nb(1+x)+Qb^2(1+x)^2+Rb^3(1+x)^3+Sb^4(1+x)^4+\&c$$

Now we know from (3) the first and second terms in the development of each of the binomials $(1+x)^1$, $(1+x)^2$, $(1+x)^3$, $(1+x)^4$, &c.; and collecting all the first terms together and all the second terms together, we have for the above development the series

$$(1+nb+Qb^2+Rb^3+Sb^4+\ \&c.)+(nb+2Qb^2+3Rb^3+4Sb^4+\ \&c.)x+\ \&c.$$

But $1+b(1+x)$ is the same as $(1+b)+bx$, and by (4) the development of $\{(1+b)+bx\}^n$ is the series

$$(1+b)^n+n(1+b)^{n-1}bx+\ \&c.$$

Of this series, as also of that before obtained for *the same expression*, only two terms, the first and second, are actually exhibited; but as the two series must be equal, whatever be the value of x, the coefficients of the like powers x must be equal (47).

$$\therefore\ n(1+b)^{n-1}b=nb+2Qb^2+3Rb^3+4Sb^4+\ \&c.$$

$$\therefore\ n(1+b)^{n-1}\ =n\ +2Qb\ +3Rb^2+4Sb^3+\ \&c.$$

Multiply each side by $1+b$, we shall then have

$$n(1+b)^n=n+(2Q+n)b+(3R+2Q)b^2+(4S+3R)b^3+\ \&c.$$

But (3), $n(1+b)^n=n+n^2b+nQb^2+nRb^3+\ \&c.$

The two series on the right must therefore be equal, whatever be the value of b; consequently (47).

$$2Q+n=n^2\quad\therefore\quad Q=\frac{n(n-1)}{2}$$

$$3R+2Q=nQ\quad\therefore\quad R=\frac{n(n-1)(n-2)}{2\cdot3}$$

$$4S+3R=nR\quad\therefore\quad S=\frac{n(n-1)(n-2)(n-3)}{2\cdot3\cdot4}$$

&c. &c.

where the *law* of the coefficients is sufficiently obvious. It follows therefore that the development (4) is

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2\cdot3}a^{n-3}x^3+\ \&c.\ \text{(A)}$$

51. And this development is the BINOMIAL THEOREM. You see that each term is related to that which immediately precedes it by a uniform law, not only as respects the coefficients, but as respects the letters a, x; thus, disregarding the coefficients, and attending only to the letters a, x, the first term is a^n, and every subsequent term (omit-

ting coefficients) is found by multiplying the immediately preceding term by $\frac{x}{a}$, the powers of a regularly *decreasing* by unity, and the powers of x as regularly *increasing* by unity, so that *the sum of the exponents is always* n; the law of the coefficients is abundantly obvious from the expressions for P, Q, R, &c., deduced above; the coefficient of the second term is n, or $\frac{n}{1}$, and every subsequent coefficient is obtained from that immediately preceding it, by introducing an additional factor into both numerator and denominator, the factors of the numerators uniformly *decreasing* by 1, and the factors of the denominators as uniformly *increasing* by 1.

You perceive that the additional factor here spoken of, which enters the numerator, is no other than the exponent of a in the preceding term, and that the additional factor which enters the denominator is the exponent of x in the same preceding term, *increased by* 1. Hence, from having presented to you only a single isolated term of the entire development, you may, without any other aid, discover: 1. To what power of $a+x$ that term belongs; 2. What place in the entire development the given term occupies; and 3. May write down all the subsequent terms of that development, and, if need be, all the preceding terms. Thus, suppose the term $56a^5x^3$ were presented to us as one of the terms of some power of $a+x$.

1. As the sum of the exponents is 8, we conclude at once that the power is the *eighth*, so that the development is that of $(a+x)^8$. 2. As the exponent of x in any term is always 1 less than the number which marks the place of that term, we conclude further that $56a^5x^3$ is the *fourth* term of the development. 3. To find the coefficient of the next or *fifth* term, we must multiply 56 by the additional factor 5, and then divide by 4, or, which is better, divide by 4 first, and then multiply by 5; the result is 70; hence the complete fifth term is $70a^4x^4$. And, in a similar way, from this we could easily discover the sixth term, and so on to the end. As to the terms preceding the given fourth, they are got by writing down the first a^8, and thence deducing the others, as above; the complete development would be found to be

$$(a+x)^8 = a^8 + 8a^7x + 28a^6x^2 + 56a^5x^3 + 70a^4x^4 + 56a^3x^5 + 28a^2x^6 + 8ax^7 + x^8$$

You cannot fail to observe here that the coefficients form the same series of numbers, whether you write them in direct or in reverse order. Such is always the case, whatever positive integer the exponent may be. The *coefficients*—from the very principle of their formation—are never affected by the particular letters of the binomial; those above would, of course, have remained the same had the left-hand expression been $(p+q)^8$, or $(x+y)^8$, or $(x+a)^8$, &c. If it had been $(x+a)^8$, the development would have been

$$(x+a)^8 = x^8 + 8x^7a + 28x^6a^2 + 56x^5a^3 + 70x^4a^4 + 56x^3a^5 + 28x^2a^6 + 8xa^7 + a^8$$

where a and x have merely changed places on both sides of the identity, so that one development being reversed, we have the other. And it is plain that such is always the case, and, therefore, that the coefficients, when written in reverse order, form the same series as when they are written in direct order.

When the exponent is odd, the number of terms is even, and when the exponent is even the number of terms is odd, thus, $(a+1)^1=a+x$, $(a+x)^2=a^2+2ax+x^2$, $(a+x)^3=a^3+3a^2x+3ax^2+a^3$, and so on. Hence, in the case of an even exponent, there is always a middle term; in the case of an odd exponent, there are two middle terms; hence, when the exponent is even, the coefficients need be computed only up to the middle term inclusive, and then for the following terms it is sufficient to copy in reverse order the coefficients preceding that middle term. When the exponent is odd, the coefficients need be computed only up to the first of the two middle terms, and the subsequent coefficients will be those already written, put down in reverse order.

The entire number of terms for all positive integral values of n is $n+1$, so that when n is even we need compute only the first $\frac{1}{2}n+1$ coefficients; and if n be odd, only the first $\frac{1}{2}(n+1)$.

If n be a negative number, or a fraction, then it would be absurd to speak about a middle term, or a pair of middle terms, in the development; for, as the general development (A) shows, the series could never come to an end; the factors in the numerators, when n is negative, would themselves all be negative, and would become greater and greater. And when n is a fraction, it is plain that no factor could ever become *zero*, so that a last or final term could never be reached; it is only when n is positive and integral that $(n-1)$, $(n-2)$, $(n-3)$, &c., could ever at length become zero, and thus put an end to the series; the term involving $(n-n)$ would, of course, be zero, and that which precedes this—which is the $n+1$th term—would be the final term.

We shall separate the practical applications of the binomial theorem into two cases: first, when the exponent is a whole number, positive or negative; and secondly, when it is a fraction.

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52. I. *To find the development of* $(a+x)^n$ *when* n *is a whole number, either positive or negative.*

RULE. Write down the first term of the development, a^n; the coefficient in this term is, of course, 1; then the second and each subsequent term is found thus:—

Multiply the coefficient in the term last found by the exponent of a in that term; the product, divided by the exponent of x in the same term increased by 1 (or by the *number* which marks the *place* of that term), will give the coefficient of the next term. And with respect to the letters, the powers of a are to *diminish*, and those of x to *increase*, by unity, in each successive term.

Observe, 1. That when n is positive, the coefficients are to be computed only up to the middle of the development, and are then to be repeated in reverse order.

2. When one of the two terms of the binomial is negative, all the odd powers of that term will be negative.

(1) Required the development of $(a+x)^7$.

1st term, 2nd term, 3rd term, 4th term,

$$a^7, \quad 7a^6x, \quad \frac{7\cdot 6}{2}a^5x^2=21a^5x^2, \quad \frac{21\cdot 5}{3}a^4x^3=35a^4x^3$$

As the number of terms is $7+1=8$, we have now reached the first of the pair of middle terms, and may therefore complete the development without any further calculation,

$$\therefore\ (a+x)^7=a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7$$

(2) Required the development of $(a-x)^6$.

1st term, 2nd term, 3rd term, 4th term,

$$a^6 \quad -6a^5x \quad \frac{6\cdot 5}{2}a^4x^2=15a^4x^2, \quad -\frac{15\cdot 4}{3}a^3x^3=-20a^3x^3$$

As the number of terms is $6+1=7$, the fourth is the middle term, therefore for the complete development we have

$$(a-x)^6=a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$$

(3) Required the development of $(a+x)^{-3}$, or $\dfrac{1}{(a+x)^3}$.

1st term, 2nd term, 3rd term, 4th term,

$$a^{-3}, \quad -3a^{-4}x, \quad \frac{-3\times -4}{2}a^{-5}x^2=6a^{-5}x^2, \quad \frac{6\times -5}{3}a^{-6}x^3=-10a^{-6}x^3, \text{ \&c.}$$

$$\therefore\ (a+x)^{-3}=a^{-3}-3a^{-4}x+6a^{-5}x^2-10a^{-6}x^3+15a^{-7}x^4-21a^{-8}x^5+\text{ \&c.}$$

$$=\frac{1}{a^3}-\frac{3}{a^4}x+\frac{6}{a^5}x^2-\frac{10}{a^6}x^3+\frac{15}{a^7}x^4-\frac{21}{a^8}x^5+\text{ \&c.}$$

$$=\frac{1}{a^3}(1-\frac{3}{a}x+\frac{6}{a^2}x^2-\frac{10}{a^3}x^3+\frac{15}{a^4}x^4-\frac{21}{a^5}x^5+\text{ \&c.})$$

a series which may be indefinitely continued, as it never comes spontaneously to an end. If the expression had been $(a+x)^{-2}$, or $\dfrac{1}{(a+x)^2}$, we should, in like manner, have had

$$(a+x)^{-2}=\frac{1}{a^2}(1-\frac{2}{a}x+\frac{3}{a^2}x^2-\frac{4}{a^3}x^3+\frac{5}{a^4}x^4-\frac{6}{a^5}x^5+\text{\&c.})$$

[In such cases as these it is of course impossible to exhibit on the right the complete algebraic equivalent of the expression on the left; we can only present a few of the leading terms of the development; what is wanted to make up the complete equivalent is concealed under the "&c." Nor can we make use of such an infinite series to compute the value of the undeveloped expression for particular numerical values of a and x, unless we know that, after a certain finite number of terms of the series, the supplemental value, under the "&c.", becomes practically of no consideration, becoming less and less as the number of terms increases, and ultimately vanishing.]

(4) Required the development of $(a+2x)^5$.

1st term, 2nd term, 3rd term,

$$a^5, \quad 5a^4(2x), \quad \frac{5\times 4}{2}a^3(2x)^2=10a^3(2x)^2$$

Hence the complete development is

$$(a+2x)^5 = a^5 + 5a^4(2x) + 10a^3(2x)^2 + 10a^2(2x)^3 + 5a(2x)^4 + (2x)^5$$
$$= a^5 + 10a^4x + 40a^3x^2 + 80a^2x^3 + 80ax^4 + 32x^5$$

(5) Required the development of $(a-3x)^5$.

1st term,	2nd term,	3rd term,
a^5,	$-5a^4(3x)$,	$10a^3(3x)^2$

$$\therefore\ (a-3x)^5 = a^5 - 5a^4(3x) + 10a^3(3x)^2 - 10a^2(3x)^3 + 5a(3x)^4 - (3x)^5$$
$$= a^5 - 15a^4x + 90a^3x^2 - 270a^2x^3 + 405ax^4 - 243x^5$$

(6) Required the development of $(2ax-3y)^5$.

1st term,	2nd term,	3rd term,
$(2ax)^5$,	$-5(2ax)^4(3y)$,	$10(2ax)^3(3y)^2$

$$\therefore\ (2ax-3y)^5 = (2ax)^5 - 5(2ax)^4 3y + 10(2ax)^3(3y)^2 - 10(2ax)^2(3y)^3 + 5(2ax)(3y)^4 - (3y)^5$$
$$= 32a^5x^5 - 240a^4x^4y + 720a^3x^3y^2 - 1080a^2x^2y^3 + 810axy^4 - 243y^5$$

Examples for Exercise.

(1) Required the development of $(a+x)^8$.

(2) What is the seventh power of $a-x$?

(3) Required the seventh power of $x+2y$.

(4) Required the development of $\frac{1}{(1+x^2)^3}$.

(5) Required the development of $(a+2x)^{-3}$.

(6) Prove that when n is a positive whole number,

$$1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{2\cdot3} + \ldots\ldots = 2^n$$

and $$1 - n + \frac{n(n-1)}{2} - \frac{n(n-1)(n-2)}{2\cdot3} + \ldots\ldots = 0^*$$

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53. II. *To find the development of* $(a+x)^{\frac{m}{n}}$, *the exponent being a fraction either positive or negative.*

Rule. Proceed exactly as in the former rule, taking the given fractional exponent for $\frac{m}{n}$, and then writing out the development, always using a and x. When this is done, then substitute for a and x the particular values for them given in the example.

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* The student is here cautioned against falling into the error—countenanced by the books on Algebra—that these equations hold *whatever be the value of* n. The error would be similar to that committed by supposing the infinite series $1 + r + r^2 + r^3 + r^4 + \ldots\ldots$ to be equal to $\frac{1}{1-r}$, whatever be the value of r. If this fraction be developed by actual division, then, at whatever term—calling it the nth term—we stop, there will always be a *remainder;* which, connected with the quotient, $1-r$ being placed under it as denominator, will be the fractional correction of that quotient; if n be infinite, this fraction will be the proper correction of the infinite series. (See the remarks at page 90).

(1) Required the development of $(b^2-x)^{\frac{3}{4}}$.

$$(a+x)^{\frac{3}{4}}=a^{\frac{3}{4}}+\frac{3}{4}a^{-\frac{1}{4}}x-\frac{3}{4\cdot8}a^{-\frac{5}{4}}x^2+\frac{3\cdot5}{4\cdot8\cdot12}a^{-\frac{9}{4}}x^3-\text{ \&c.}$$

Substituting b^2 for a, and $-x$ for x,

$$\therefore\ (b^2-x)^{\frac{3}{4}}=b^{\frac{3}{2}}-\frac{3}{4}b^{-\frac{1}{2}}x-\frac{3}{4\cdot8}b^{-\frac{5}{2}}x^2-\frac{3\cdot5}{4\cdot8\cdot12}b^{-\frac{9}{2}}x^3-\text{ \&c.}$$

(2) Required the development of $\frac{1}{\sqrt{(1-x)}}$ or $(1-x)^{-\frac{1}{2}}$.

$$(a+x)^{-\frac{1}{2}}=a^{-\frac{1}{2}}-\frac{1}{2}a^{-\frac{3}{2}}x+\frac{3}{2\cdot4}a^{-\frac{5}{2}}x^2-\frac{3\cdot5}{2\cdot4\cdot6}a^{-\frac{7}{2}}x^3+\text{ \&c.}$$

$$\therefore\ (1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{3}{2\cdot4}x^2+\frac{3\cdot5}{2\cdot4\cdot6}x^3+\text{ \&c.}$$

Examples for Exercise.

(1) Required the development of $(a^2+x^2)^{\frac{1}{2}}$.

(2) Required the development of $(a^2-x^2)^{\frac{1}{2}}$.

(3) Develope $(a^3-b^3)^{\frac{1}{3}}$ in a series.

(4) Develope $(a^5+x^5)^{\frac{1}{5}}$ in a series.

(5) Develope $(b^2+x)^{-\frac{1}{2}}$.

(6) Find a series for the cube root of 9; that is, develope $(8+1)^{\frac{1}{3}}$ by the binomial theorem, and thence deduce the root to eight or ten places of decimals.

54. [The binomial theorem can be applied to the purposes of arithmetical calculation, as in this sixth example, only when the particular case of the series is such that the more of its leading terms we add together the nearer we approach to a fixed and finite number, equivalent to the entire sum of the series taken in all its infinitude of terms. In order to this, the terms of the series—at least, after a certain number—must continually diminish and tend towards zero, as the number of terms tends to infinity; the term infinitely remote being infinitely small, or zero. A series thus admitting of a finite value, and whose terms, therefore, continually diminish down to zero, is called a *converging* series, while a series whose terms, on the contrary, continually increase, is called a *diverging* series; such a series has, of course, no finite sum, and cannot be employed in calculation. A series whose terms neither increase nor decrease, but remain constant, as the series $1-1+1-1+1-\ldots\ldots$, is called a *neutral* series.

The general series determined in the foregoing examples all become useless for the purposes of calculation, except for certain particular values of the symbols; thus, if we take the series

$$(a+x)^{-2}=\frac{1}{a^2}\left(1-\frac{2}{a}x+\frac{3}{a^2}x^2-\frac{4}{a^3}x^3+\frac{5}{a^4}x^4-\frac{6}{a^5}x^5+\text{ \&c.}\right)\ldots\ldots(1)$$

at page 88, and put a and x each equal to 1, we have

$$(1+1)^{-2}=\frac{1}{\sqrt{2}}=1-2+3-4+5-6+\text{ \&c.}\ldots\ldots(2)$$

But to say that $\frac{1}{\sqrt{2}}$ is equal to the series of natural numbers, taken with alternate signs and extended to infinity, is palpably absurd; the series is divergent, and has no finite sum. The fact is, that as the terms do not ultimately vanish, as in a convergent series, there is a supplementary correction of the series, included under the "&c." in (1), which, upon putting in it 1 for a and x, would supply the correction of the series (2), fitted to render that series, stop at what term we may, the strict equivalent of $\frac{1}{\sqrt{2}}$. And such a correction is *always* necessary whenever the series is divergent; but, in general, it is not to be determined.

The more convergent a series is, that is, the more rapidly its terms approach in value to zero, the fewer of those terms require to be added to furnish an approximation to the sum, sufficiently near for practical purposes. It is pretty obvious that a series is always convergent when any term, however remote, divided by the preceding one, is less than 1. The harmonical series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$, is not convergent, though its terms diminish; it does not fulfil this test, for $\frac{1}{n}\div\frac{1}{n-1}=1-\frac{1}{n}$ is not less than 1 when n is ∞. The sum of this series is infinite: on the contrary, the geometrical series

$$\frac{1}{1-r}=1+r+r^2+r^3+r^4+\ldots\ldots$$

is always convergent when $r \angle 1$; for $r^n \div r^{n-1}=r$, whatever be the value of n; but if $r \,7\, 1$, then the series (without a supplementary correction) is no longer the equivalent of $\frac{1}{1-r}$.

In approximating to the value of a converging infinite series by actually summing up its leading terms, it is useful to be able to estimate the amount of error committed by disregarding all the terms which follow those added up. When the terms of the converging series are alternately + and −, this is easily done; thus, let the series be

$$S=a-b+c-d+e-f+g-h+\ldots\ldots$$

then if we stop at a negative term, as f, we shall neglect the quantity

$$(g-h)+(k-l)+(m-n)+\ldots.$$

of which all the terms are *positive*, because g, h, k, &c., continually diminish,

$$\therefore\ a-b+c-d+e-f \angle S$$

but if we stop at a positive term, g, then we shall omit the quantity

$$-(h-k)-(l-m)-\ldots.$$

of which all the terms are negative,

$$\therefore\ a-b+c-d+e-f+g \,7\, S$$

It follows, therefore, that at whatever term we stop, the error committed is always numerically less than the term immediately following; the error is in excess if we stop at a positive term, and in defect if we stop at a negative term.

When the terms of the converging series have all the same sign, a limit to the amount of error is found by taking a geometrical series of which the first term is the same as that immediately following the term at which we stop, and in which the ratio is equal to the *greatest* ratio that can have place between two consecutive terms of the suppressed series. The sum of this geometrical progression must, of course, exceed the sum of the neglected terms of the proposed series, and will, therefore, make known a limit which the error committed cannot reach.

For a complete examination of the peculiarities of infinite series, the advanced student may consult a paper on the subject by the author of this work, in Vol. IX. of the Transactions of the Cambridge Philosophical Society.*]

INVESTIGATION OF THE EXPONENTIAL THEOREM.

55. The expression a^x, where the unknown, or perfectly arbitrary symbol x, enters as an exponent, is called an exponential quantity; the object of the exponential theorem is to develop this expression in a series proceeding according to the positive integral powers of x. That it really admits of such development is shown as follows:—

Write $1+b$ for a, that is, call $a-1$, b, so that the symbol a may have a binomial form; then by the binomial theorem we have, for a^x, or $(1+b)^x$, the development

$$a^x = 1 + xb + \frac{x(x-1)}{2}b^2 + \frac{x(x-1)(x-2)}{2\cdot 3}b^3 + \frac{x(x-1)(x-2)(x-3)}{2\cdot 3\cdot 4}b^4 + \&c.$$

Now if the factors entering the numerators of these terms were actually multiplied together, the result would evidently be a series of terms such as $Ax + Bx^2 + Cx^3 + \&c.$, which is sufficient to show that the proposed form of development is possible.

Without attempting to execute in full the multiplications here spoken of, it is plain that the first term in the arranged development, after the 1, which first term we have represented by Ax, would be

$$Ax = (b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.)\,x$$

From what is now shown we are then justified in assuming

$$a^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c. \ldots\ldots (1)$$

$$\therefore\ a^y = 1 + Ay + By^2 + Cy^3 + Dy^4 + \&c. \ldots\ldots (2)$$

And also $a^{x+y} = 1 + A(x+y) + B(x+y)^2 + C(x+y)^3 + D(x+y)^4 + \&c.$

The last two identities must of course have place because the first is an identity; that is, it is an equation which holds *whatever* be put

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[*] For methods of summing a great variety of converging infinite series, reference may be made to the author's Elements of Algebra, fourth edition.

at page 88, and put a and x each equal to 1, we have

$$(1+1)^{-2}=\frac{1}{\sqrt{2}}=1-2+3-4+5-6+\text{ \&c.}\ldots\ldots(2)$$

But to say that $\frac{1}{\sqrt{2}}$ is equal to the series of natural numbers, taken with alternate signs and extended to infinity, is palpably absurd; the series is divergent, and has no finite sum. The fact is, that as the terms do not ultimately vanish, as in a convergent series, there is a supplementary correction of the series, included under the "&c." in (1), which, upon putting in it 1 for a and x, would supply the correction of the series (2), fitted to render that series, stop at what term we may, the strict equivalent of $\frac{1}{\sqrt{2}}$. And such a correction is *always* necessary whenever the series is divergent; but, in general, it is not to be determined.

The more convergent a series is, that is, the more rapidly its terms approach in value to zero, the fewer of those terms require to be added to furnish an approximation to the sum, sufficiently near for practical purposes. It is pretty obvious that a series is always convergent when any term, however remote, divided by the preceding one, is less than 1. The harmonical series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$, is not convergent, though its terms diminish; it does not fulfil this test, for $\frac{1}{n}\div\frac{1}{n-1}=1-\frac{1}{n}$ is not less than 1 when n is ∞. The sum of this series is infinite: on the contrary, the geometrical series

$$\frac{1}{1-r}=1+r+r^2+r^3+r^4+\ldots\ldots$$

is always convergent when $r<1$; for $r^n\div r^{n-1}=r$, whatever be the value of n; but if $r>1$, then the series (without a supplementary correction) is no longer the equivalent of $\frac{1}{1-r}$.

In approximating to the value of a converging infinite series by actually summing up its leading terms, it is useful to be able to estimate the amount of error committed by disregarding all the terms which follow those added up. When the terms of the converging series are alternately + and −, this is easily done; thus, let the series be

$$S=a-b+c-d+e-f+g-h+\ldots\ldots$$

then if we stop at a negative term, as f, we shall neglect the quantity

$$(g-h)+(k-l)+(m-n)+\ldots\ldots$$

of which all the terms are *positive*, because g, h, k, &c., continually diminish,

$$\therefore\ a-b+c-d+e-f<S$$

but if we stop at a positive term, g, then we shall omit the quantity

$$-(h-k)-(l-m)-\ldots\ldots$$

of which all the terms are negative,

$$\therefore\ a-b+c-d+e-f+g>S$$

THE THEORY OF LOGARITHMS.

56. The invention of Logarithms is due to Lord Napier, of Merchistoun, in Scotland. This distinguished person being much devoted to matters of practical calculation, and feeling the irksomeness of the common operations of multiplication and division whenever large numbers were concerned, at length conceived a plan whereby these lengthy processes may be replaced by the simpler operations of addition and subtraction. How this is brought about may be explained as follows:

1. Every positive number may be regarded as produced from some power, whole or fractional, positive or negative, of any assumed base-number or root, provided that root be positive, and other than unity. For if a stand for the assumed base, and n for any positive number whatever, then the solution of the equation $a^x=n$, that is, the determination of x in this equation, will make known the proper exponent to be put over the given number a, so that the power a^x may be equal to the given number n.

2. It will be shown presently that the value of x in the equation $a^x=n$, where a is any positive number other than unit, chosen at pleasure, and n any proposed positive number whatever, can always be found by aid of the exponential theorem; in other words, that any positive number n can always be represented under the form a^x, a being any chosen number, except unit, but which when once chosen remains invariable. The suitable exponent x for any given number n is called the *logarithm* of n, to the assumed base a, and the form $x=\log n$ stands for the words "x is the logarithm of n." A table in which the proper values of x are inserted against the respective numbers 1, 2, 3, &c., up to any proposed limit, is called a *table of logarithms* of the numbers 1, 2, 3, &c., up to that limit; so that a table of *logarithms* is in reality nothing more than a table of *exponents*, it being understood that these exponents apply to some fixed base of which the value is declared. It is found that 10, for several reasons, is the number most convenient for a base, and accordingly our modern tables of logarithms or exponents are always constructed in reference to this base. In such tables, therefore, we may expect to find, already computed for us, the proper values of x, fulfilling the conditions $10^x=1$, $10^x=2$, $10^x=3$, $10^x=4$, &c., that is, we may expect to find, computed for the base 10, $\log 1$, $\log 2$, $\log 3$, $\log 4$, &c. Of course $\log 1$ is always 0, whatever be the base; because, without any restriction as to the value of a, we know that $a^0=1$ (see page 34). The following theorems will suffice to show the practical use of a table of logarithms: how such a table may be constructed will be explained afterwards.

Theorem 1. If the logarithms of any two or more numbers be *added* together, the sum will be the logarithm of the *product* of those numbers.

Let $a^x=n$, and $a^{x'}=n'$; that is, let $x=\log n$, and $x'=\log n'$. Then $a^x \times a^{x'}=nn'$, that is, $a^{x+x'}=nn'$, so that $x+x'=\log nn'$; but $x+x'=\log n+\log n'$,

$$\therefore \log n+\log n'=\log nn'$$

In like manner,

$$\log nn'+\log n''=\log nn'n''=\log n+\log n'+\log n''$$

and so on. Hence the *sum* of the logarithms of any number of factors is equal to the logarithm of their *product*; and consequently, m times the logarithm of any number is equal to the logarithm of the mth power of that number; that is, m being a positive whole number, $m \log n = \log n^m$.

Theorem 2. The *difference* of the logarithms of any two numbers is the logarithm of the *quotient* of those numbers.

Let $a^x = n$, and $a^{x'} = n'$; then $a^x \div a^{x'} = a^{x-x'} = \frac{n}{n'}$, $\therefore$ $x - x' = \log\frac{n}{n'}$, that is, $\log n - \log n' = \log\frac{n}{n'}$.

Theorem 3. The mth part of the logarithm of any number is equal to the logarithm of its mth root.

For let $a^x = n$ $\therefore$ $a^{\frac{x}{m}} = n^{\frac{1}{m}}$ $\therefore$ $\frac{x}{m} = \log n^{\frac{1}{m}}$, that is, $\frac{\log n}{m} = \log \sqrt[m]{n}$. And generally whatever be the exponent m, whether positive or negative, whole or fractional, $m \log n$ is always the log of n^m; for since $a^x = n$ $\therefore$ $a^{mx} = n^m$ $\therefore$ $mx = \log n^m$, that is, $m \log n = \log n^m$.

These theorems sufficiently show the great use of a table of logarithms in abridging complicated arithmetical operations. If we have to find a *product*, we take from the table the logarithms of the factors, *add* them together, and thus get a log, and opposite to this log in the table is the product sought. If we have to find a *power*, we take from the table the log of the root, and multiply it by the exponent; the product is a log, against which, in the table, is the power sought, whether that power be whole or fractional (*Theo.* 1 and 3).

If we have to find the quotient of two numbers, we take from the table the logs of dividend and divisor, subtract the latter from the former, and for the difference get a log, opposite to which, in the table, is the quotient sought (*Theo.* 2).

Investigation of series for the construction of Logarithms.

57. Let a represent the base of the system of logarithms to be constructed; then, putting N for any positive number, we have to find a convergent series for x, in terms of a and N, from the equation

$$a^x = N. \text{ This gives } a^{xy} = N^y \; \therefore \text{ (Exp. Theorem.)}$$

$$1 + Axy + \frac{A^2}{2}x^2y^2 + \text{\&c.} = 1 + A'y + \frac{A'^2}{2}y^2 + \text{\&c.}$$

two series which remain equal whatever be the value of y,

$$\therefore \text{(47)}, \; Ax = A' \; \therefore \; x = \frac{A'}{A} = \frac{(N-1) - \frac{1}{2}(N-1)^2 + \frac{1}{3}(N-1)^3 - \text{\&c.}}{(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \text{\&c.}}$$

(by eq. 5, page 93),

that is, substituting $(1+n)$ for N, and putting M for $\frac{1}{(a-1) - \frac{1}{2}(a-1)^2 + \text{\&c.}}$

$$\log(1+n)=M\left(n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\text{ \&c.}\right)\ldots.(\text{A})$$

The multiplier $M=\frac{1}{A}$, is called the *modulus* of the particular system of logarithms; you see it depends for value entirely upon the particular base a that has been selected. If a were chosen so that M might be 1, the general series above for the log of any number $1+n$ would be simply $n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\text{\&c.}$ To secure this simplicity of form, Napier made *unity* the modulus of his system, and then determined the corresponding base from the condition $A=1$.

58. And here it may be well to suspend the investigation for a moment, to draw your special attention to two circumstances which must be always borne in mind.

1. That whatever the base a may be, the logarithm of that base, in the system to which it belongs, is always 1; for $a^1=a$, $\therefore 1=\log a$.

2. The logarithms of *the same number*, in any two systems, are to one another as the moduli of those systems; for the right-hand member of (A), provided n remain unaltered, would differ from its present value if the system were changed from that in which the modulus is M, to another in which the modulus is M', only in having M replaced by M', so that the *ratio* of the two expressions for $\log(1+n)$ would be $\frac{M}{M'}$.

Let now e stand for the value of a, which satisfies the equation $A=1$; then, by the exponential theorem,

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2\cdot3}+\frac{x^4}{2\cdot3\cdot4}+\text{ \&c.}$$

so that making $x=1$, we readily find for the Napierian base e, the value.

$$e=1+1+\frac{1}{2}+\frac{1}{2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ \ldots=2\cdot718281828459$$

On account of the rapidity with which this series converges, a comparatively small number of the leading terms will suffice to determine e to several places of decimals; thus, taking only ten of the terms $2+\frac{1}{2}+\frac{1}{2\cdot3}+\ldots.$, and computing as in the margin, we get e true to seven decimals.

2	1
3	·5
4	·1666667
5	416667
6	83333
7	13889
8	1984
9	248
10	27
	3
	2·7182818

Resuming now the general series for $\log(1+n)$, and leaving the base of the system, and consequently the modulus $\frac{1}{A}$, or M, open to any hypothesis, as to value, that may be hereafter introduced, we have, by taking n first positive and then negative,

$$\log(1+n)=M\left(n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\text{\&c.}\right)\ldots.(1)$$

$$\log(1-n)=M\left(-n-\frac{1}{2}n^2-\frac{1}{3}n^3-\frac{1}{4}n^4-\text{\&c.}\right)\ldots.(2)$$

Subtracting, $\log(1+n)-\log(1-n)$, that is, $\log\dfrac{1+n}{1-n}=$

$$2M\left(n+\frac{1}{3}n^3+\frac{1}{5}n^5+\&c.\right)\ldots\ldots(3)$$

59. These series are of but little direct importance in computing a table of logarithms, on account of their slow convergency, except when n is a small fraction. But, as we are dealing all along with identities, we are at liberty, at any stage of the investigation, to put anything we please for n. Now it is evidently desirable that we should obtain a rapidly-converging series for $\log\dfrac{p+1}{p}$, without restricting p to small values; because, from such a series for $\log(p+1)-\log p$, we could then calculate the logs of all the numbers 1, 2, 3, 4, &c., in succession. For $p=1$, $\log p$ is known, since the proper exponent x, for $a^x=1$, is $x=0$; and from thus having $\log p$ in the simplest case, we could thence get $\log(p+1)$ or $\log 2$, thence $\log 3$, and so on. Put therefore $\dfrac{1+n}{1-n}=\dfrac{p+1}{p}$, whence $n=\dfrac{1}{2p+1}$, and we have, (3), by transposing $\log p$ to the other side,

$$\log(p+1)=2M\left(\frac{1}{2p+1}+\frac{1}{3(2p+1)^3}+\frac{1}{5(2p+1)^5}+\&c.\right)+\log p$$

a series which increases in convergency as p becomes greater.

[It must not be supposed that, in this last step, we have replaced the series (3) by another, deduced from it, which is more convergent *for the same values*. We can never increase the convergency of a series, in this sense, by any substitutions or transformations; we have merely replaced n by $\dfrac{1}{2p+1}$; and now, making $p=1, 2, 3$, &c., successively, we, of course, get no greater convergency than (3) affords by making $n=\dfrac{1}{3}, \dfrac{1}{5}, \dfrac{1}{7}$, &c. The advantage of the change consists in replacing the fraction $\dfrac{1+n}{1-n}$ by the equal and much more suitable fraction $\dfrac{p+1}{p}$.]

Application of the foregoing series to the computation of the Napierian logarithms of the natural numbers.

60. Putting $p=1, 2, 3$, &c., in succession, the series just deduced gives the following numerical results, remembering that in the Napierian system $M=1$.

$$\text{Nap. }\log 2=2\left(\frac{1}{3}+\frac{1}{3\cdot 3^3}+\frac{1}{5\cdot 3^5}+\&c.\right)\ldots\ldots\ldots\ldots=\ \cdot 6931472$$

$$\log 3=2\left(\frac{1}{5}+\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}+\&c.\right)+\log 2\ldots\ldots=1\cdot 0986123$$

$$\log 4=2\log 2\ (\textit{Theorem } 1)\ \ldots\ldots\ldots\ldots\ldots\ldots=1\cdot 3862944$$

$$\log 5=2\left(\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\text{ \&c.}\right)+\log 4\;\ldots\ldots=1\cdot 6094379$$

$$\vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots$$

$$\log 10=\log 2+\log 5\ (\textit{Theorem } 1)\ldots\ldots\ldots\ldots=2\cdot 3025851$$

This specimen is sufficient to show how Napierian logarithms *may* be constructed, but much more commodious series may be obtained for the logarithms of numbers a little advanced in the numerical scale; thus, if instead of putting $\frac{1}{2p+1}$ for n, in (3), we had put $\frac{2}{p^3-3p}$, we should have had

$$\frac{1+n}{1-n}=\left(1+\frac{2}{p^3-3p}\right)\div\left(1-\frac{2}{p^3-3p}\right)=\frac{p^3-3p+2}{p^3-3p-2}=\frac{(p-1)^2(p+2)}{(p+1)^2(p-2)}$$

and thence,

$$2\log(p-1)+\log(p+2)-2\log(p+1)-\log(p-2)=$$
$$2\left\{\frac{2}{p^3-3p}+\frac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\frac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+\text{ \&c. }\right\}$$

and therefore by transposition,

$$\log(p+2)=\log(p-2)+2\log(p+1)-2\log(p-1)+$$
$$2\left\{\frac{2}{p^3-3p}+\frac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\frac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+\text{ \&c. }\right\}$$

a series which converges very rapidly for all values of p above 3, and which may therefore be employed with advantage when the logs, up to log 5 inclusive, have been found. This formula is due to M. Borda. Although every individual formula for the construction of a table of logarithms is simpler to the eye when the logarithms are to be those of Napier, than when any other system is adopted, yet a table constructed to the base $a=10$, instead of Napier's base $a=e=2\cdot 7182818\ldots\ldots$ is of much easier formation and of far greater practical use, as will be shown presently.

61. But before we can compute logs to the base $a=10$, we must know the corresponding value of the modulus M, which, as we have seen, in Napier's system, is simply 1. Logarithms of the same number in any two different systems are as the moduli in those systems (p. 96); therefore, distinguishing Napier's logs thus, $\log_e$, and the common (or Briggs's) logarithms, in which the base is 10, thus, $\log_{10}$, we have, by writing M for the modulus in the latter system,

$$\frac{M}{1}=\frac{\log_{10}10}{\log_e 10}=\frac{1}{2\cdot 3025851}=\cdot 43429448$$

We thus have the constant multiplier M, which must be applied to the foregoing series when the base of the system of logarithms is to be 10; the log of the *base*—taken in the system to which the base belongs—is, of course always 1 (p. 96). By extending the decimals in $\log_e 10$, found above, we have for the modulus M, the value $M=\cdot 43429448190325\ldots\ldots$ We shall now proceed to show how a table of common, or Briggs's, logarithms may be computed; they are called Briggs's logs because Briggs urged upon Napier the advantages that would arise from changing the base of the system from 2·71828....to 10, although it is known that Napier himself had already had this change in contemplation. But Briggs executed the calculations on the new system.

[As we are about to show the construction of logs to the base 10, it is sufficient for our purpose to apply the general theorem at the bottom of last page to the particular case before us, and thus to write

$$\frac{1}{A}=\frac{M}{1}=\frac{\log_{10}10}{\log_e 10},\ \text{or}\ A=\frac{\log_e 10}{\log_{10}10}=2\cdot3025851$$

But if, without assuming any particular value for the base, we represent it generally by a, then the theorem is

$$A=\frac{\log_e a}{\log_a a}=\log_e a\text{; because }\log_a a=1\ \ (\textit{Theo.}\ 1,\ \text{p. } 96)$$

Hence, writing this expression for A in the exponential theorem (p. 93), we have

$$a^x=1+x\log_e a+\frac{x^2\log_e{}^2 a}{2}+\frac{x^3\log_e{}^3 a}{2\cdot3}+\frac{x^4\log_e{}^4 a}{2\cdot3\cdot4}+\ \&\text{c.}$$

$$=1+\log_e a^x+\frac{1}{2}(\log_e a^x)^2+\frac{1}{2\cdot3}(\log_e a^x)^3+\ \&\text{c.}$$

therefore, putting y for a^x, which may be anything,

$$y=1+\log_e y+\frac{1}{2}(\log_e y)^2+\frac{1}{2\cdot3}(\log_e y)^3+\ \&\text{c.}$$

Note. $\text{Log}^n y$ means the nth power of $\log y$, and is the same as $(\log y)^n$. It may be remarked that the property $A=\log_e a$ might have been deduced at once from (5), at page 93, since the series (5) is $\log_e a$.]

Application of the foregoing series to the computation of common logarithms, or those to the base 10.

62. The general series employed in article 60, page 97, is

$$\log(p+1)=2M\left(\frac{1}{2p+1}+\frac{1}{3(2p+1)^3}+\frac{1}{5(2p+1)^5}+\ \&\text{c.}\right)+\log p$$

M being taken equal to 1, its proper value for Napier's base e. When the base is 10, then, as already shown (p. 98), the proper value of M is ·43429448; so that, for common logs, the constant multiplier $2M$ of the series, instead of 2, as in Napier's logs, will be ·86858896, and we shall have the following results by putting for p the values 1, 2, 3, &c., in succession. The distinguishing symbol $\log_{10}$ need not be preserved, it will be sufficient to introduce such distinction when logs to a base other than 10 are the subjects of inquiry.

$$\log\ 2=\cdot86858896\left(\frac{1}{3}+\frac{1}{3\cdot3^3}+\frac{1}{5\cdot3^5}+\ \&\text{c.}\right)\ldots\ldots=\cdot3010300$$

$$\log\ 3=\cdot86858896\left(\frac{1}{5}+\frac{1}{3\cdot5^3}+\frac{1}{5\cdot5^5}+\ \&\text{c.}\right)+\log 2=\cdot4771213$$

$$\log\ 4=2\log 2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots=\cdot6020600$$

$$\log\ 5 = \cdot 86858896\left(\frac{1}{9}+\frac{1}{3\cdot 9^3}+\frac{1}{5\cdot 9^5}+\ \&c.\right)+\log 4 = \cdot 6989700$$

$$\vdots$$

$$\log 10 = \log 2 + \log 5 \ldots\ldots\ldots\ldots\ldots\ldots = 1\cdot 0000000$$

And in this way may the computations be carried on to any extent.

Enough has now been said to give you a clear and satisfactory notion of the way in which logarithmic tables *may* be constructed. Neither Napier nor Briggs had the advantage of the facilities offered by algebraical series, and the existing tables, by these laborious calculators, were constructed by far more operose methods. Modern algebraists, however, have supplied formulæ in abundance for abridging the work required even by the process explained above; but it would be out of place to enter at length into these matters here. The inquiring student will find ample information respecting them in the author's "Essay on the Computation of Logarithms." It only remains for us here to explain what are the advantages of Briggs's logarithms over those first proposed by Napier.

On the peculiar advantages of the system of logarithms computed to the base 10.

63. There are two distinguishing advantages peculiar to the system of logarithms computed to the base 10, and of these, the first is, by far, the more important; they are as follows:

1. The table will be complete though the logs of only the *whole* numbers 1, 2, 3, &c., be inserted; there is no occasion for the logs of numbers involving *decimals*.

2. There is no need of inserting in the tables the *integral* part of a logarithm; the decimal part being sufficient.

These principles will be sufficiently apparent from the following illustration.

$$\log 3572 \ldots\ldots\ldots\ldots\ldots\ldots = 3\cdot 5529115$$

$$\log 357\cdot 2 = \log\frac{3572}{10} = \log 3572 - 1 = 2\cdot 5529115$$

$$\log 35\cdot 72 = \log\frac{357\cdot 2}{10} = \log 357\cdot 2 - 1 = 1\cdot 5529115$$

$$\log 3\cdot 572 = \log\frac{35\cdot 72}{10} = \log 35\cdot 72 - 1 = \cdot 5529115$$

$$\log \cdot 3572 = \log\frac{3\cdot 572}{10} = \log 3\cdot 572 - 1 = \bar{1}\cdot 5529115^{*}$$

$$\log \cdot 03572 = \log\frac{\cdot 3572}{10} = \log \cdot 3572 - 1 = \bar{2}\cdot 5529115$$

$$\log \cdot 003572 = \log\frac{\cdot 03572}{10} = \log \cdot 03572 - 1 = \bar{3}\cdot 5529115$$

&c. &c.

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[*] This is a brief way of writing $-1 + \cdot 5529115$; the minus sign being placed over the figure affected by it.

In like manner,

$$\log 3572000 = \log(3572 \times 1000) = \log 3572 + \log 10^3 = \log 3572 + 3$$
$$= 6 \cdot 5529115, \text{ \&c. \&c.}$$

This is sufficient to show, 1st. That the common log of a number always consists of the same *decimals*, however the figures of that number be separated by the decimal point, and however many zeros may follow or precede it; and, 2nd. That the integer preceding the decimals in the log always expresses the number of places which the leading significant figure of the number is from the units' place; thus, in the top number above, the leading significant figure 3 is *three* places from that of units; in the next, it is *two* places; in the fourth, it is no distance from the units' place; in the sixth, it is *two* places distant in the opposite direction, so that the integral part of the log is -2, and so on. The integral part of a log is called its *characteristic*, the decimal part its *mantissa;* to find the characteristic, we have only to count how many places the leading figure of the number is from the units' place; to make the characteristic positive when we count to the right, and negative when we count to the left. Not only is the mantissa the same for all numbers consisting of the same figures—regardless of the decimal point—but it is the same also for the reciprocals of all numbers consisting of the same figures, thus,

$$\log\frac{1}{3572} = -3 \cdot 5529115; \quad \log\frac{1}{357 \cdot 2} = -2 \cdot 5529115$$

$$\log\frac{1}{35 \cdot 72} = -1 \cdot 5529115; \quad \log\frac{1}{3 \cdot 572} = - \quad \cdot 5529115$$

$$\log\frac{1}{\cdot 3572} = -\bar{1} \cdot 5529115; \quad \log\frac{1}{\cdot 03572} = -\bar{2} \cdot 5529115; \text{ \&c. \&c.}$$

where $-\bar{1} \cdot 5529115 = 1 - \cdot 5529115$, $-\bar{2} \cdot 5529115 = 2 - \cdot 5529115$, &c.

As the characteristic, or integral part, of the common log of any number is always easily determined by inspection, the mantissa, or decimal part, is all that it is necessary to insert in the tables.

The advantages here shown to belong to the logarithms constructed to the base 10 are peculiar to that system; in every other system, each particular number would require a distinct logarithm; whereas here the logarithms of the *whole* numbers suffice for every purpose, and the decimal part only of each log is all that need be tabulated.

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64. [Before concluding this section on the theory and computation of logarithms, it may be well to give insertion to a formula of some use in the more advanced departments of analysis; we shall have occasion for it in the second volume of this Course.

If in the formula (1) page 96 we put $\frac{1}{n}$ for n, we shall have

$$\log\frac{n+1}{n} = M\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \text{\&c.}\right)$$

$$\therefore\ \log(n+1)-\log n=M\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\text{ \&c.}\right)$$

$$\text{But } \log(n+1) \qquad =M\left(n-\frac{n^2}{2}+\frac{n^3}{3}-\frac{n^4}{4}+\text{ \&c.}\right)$$

$$\text{Subtracting,} \qquad \log n=M\left\{(n-n^{-1})-\frac{1}{2}(n^2-n^{-2})+\frac{1}{3}(n^3-n^{-3})-\text{\&c.}\right\}$$

We would also strongly recommend to the student who aims at something beyond a superficial acquaintance with algebraical science, to carefully study the various expedients by which the several results in this section have been obtained. He should especially notice how the series

$$A=(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\text{ \&c.}\ldots\ldots(1)$$

originated, and by what means we found out that, when $a=10$, its value is 2·3025851. To have discovered this by *summing* the series would have been impossible, for

$$9-\frac{1}{2}(9)^2+\frac{1}{3}(9)^3-\frac{1}{4}(9)^4+\ \ldots\ldots$$

continued to infinity, has no finite sum, as is obvious. The value $A=2{\cdot}3025851$ was discovered indirectly, by taking the expression (1) above in all its integrity, and not omitting whatever may be concealed under the "&c." By taking no account of this *supplement* to the series, and considering only the series itself, for $a=10$, we should be led to the rapidly *diverging* form just exhibited, in reference to which it would be absurd to employ the term *sum*, for it has no sum. The most serious mistakes have been committed by neglecting the supplemental "&c." in the treatment of diverging series. It would be well if the "&c." were introduced only as an indication of a supplementary correction of the series, and omitted when the series *alone*, without any additive or subtractive appendage, is to be considered; the endless continuation of the terms being implied in the dots sometimes used to replace the "&c.," as in the numerical series above.*

65. There is another matter, too, to which the attention of the advanced student should be directed, namely, that in all the general developments of analysis the influence of unit-factors should be recognised whenever roots of the expressions developed are concerned. The root of a quantity has the same number of values as there are units in the denominator of the fractional exponent; and if the roots of 1 be neglected, the development may have but one value, while the undeveloped equivalent of it has several, which implies contradiction: thus,

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* Mr. De Morgan proposes to call the sum of a series its *invelopment*—a very good term—signifying that, the development of which reproduces the series. The exclusively convergent cases of a series have, in general, no algebraical invelopment, the sum, in such cases, being usually numerical only. The general algebraic invelopment includes, *of course*, the "&c.," which vanishes in the convergent cases.

in the exponential theorem, if $x=\frac{1}{m}$, then $a^{\frac{1}{m}}$ has m values, while the development appears to have but one. This defect would have been remedied, if, at the outset of the investigation, we had assumed $a=p(1+b)$, and, therefore, $a^x=p^x(1+b)^x$, and had then made $p=1$. But without encumbering our expressions with an additional symbol, we may always introduce into the final result of the investigation such a root of 1 as will cause the development to have as many values as the undeveloped form. The roots of unity will often be found very useful for this purpose; that $1^{\frac{1}{m}}$ has m values will be shown in the TRIGONOMETRY.]

COMPOUND INTEREST, ANNUITIES, REVERSIONS, &c.

66. Interest is the sum paid for the use of money—*the principal*—for any stated period. If the interest be regularly received as it becomes due, the money, or principal, is said to be put out at Simple Interest; but if, instead of being regularly received, it is added to the principal, thus increasing the sum for which interest is allowed, then the money is said to be put out at Compound Interest.

Let P = the principal put out.
n = the time, or number of years.
A = the amount in that time.
R = the amount of £1 in 1 year.
$=1+r$, r being the *interest* of £1 for 1 year.

To find the amount (A) *of a given sum* (P) *in any number* (n) *of years, at compound interest.*

The amount of £1 in 1 year is R, therefore the amount of R pounds in 1 year must be R times as much, that is, R^2 pounds; the amount of this in 1 year must be R^3 pounds, and so on; or by proportion,

$$\begin{array}{llll} 1 : R :: R & : R^2 = \text{amount in} & 2 \text{ years.} \\ 1 : R :: R^2 & : R^3 = \quad ,, & 3 \text{ years.} \\ \vdots & & \\ 1 : R :: R^{n-1} & : R^n = \quad ,, & n \text{ years.} \end{array}$$

and as the amount of P pounds must be P times as much, we have

$$A=PR^n \quad \therefore \ \log A=\log P+n\log R.$$

From this equation, any one of the four quantities A, P, R, n, may be found, when the other three are given. If it be required to find the number of years in which a principal will amount to any number of times itself, or to any proposed sum A, then putting $A=mP$, or = the proposed sum, we have

$$n=\frac{\log mP-\log P}{\log R}=\frac{\log m+\log P-\log P}{\log R}=\frac{\log m}{\log R}, \text{ or } n=\frac{\log A-\log P}{\log R}$$

67. In these formulæ and throughout the present article, n, though supposed here to be years, may represent any other regular periods of time, as half-years, quarters, months, &c.; but care must be taken

always to make r the interest of £1 for one of the n periods, and R the amount of £1 in that time. It is scarcely necessary to state that r is the rate per cent. divided by 100, that is, i being the interest of £100 for one period, r is $\frac{i}{100} = \cdot 0i$, and therefore, $R = 1 \cdot 0i$ ($\cdot 0i$ being here written for $i \times \cdot 01$).

It is also evident that the value of P, deduced from the above equation, namely, $P = \frac{A}{R^n}$, is the present value of a sum A, receivable n years to come, interest being at the rate of r for £1.

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1. What will be the amount of £325 in 4 years, at 5 per cent., compound interest?

$$P = 325,\ n = 4,\ R = 1 \cdot 05$$

By the formula, $\log A = \log P + n \log R$

$$\log 325 + 4 \log 1 \cdot 05$$

$$\begin{array}{rr} \log 1 \cdot 05 = & \cdot 0211893 \\ & 4 \\ \hline & \cdot 0847572 \\ \log 325 = & 2 \cdot 5118834 \\ \hline \log A = & 2 \cdot 5966406 \\ \hline \end{array} \quad \therefore A = 395 \cdot 0394 l. = 395 l.\ 0 s.\ 9\tfrac{1}{2} d.$$

2. How much money must be placed out at compound interest to amount to £1000 in 20 years, interest being at 5 per cent.?

$$A = 1000,\ n = 20,\ R = 1 \cdot 05$$

$$\log P = \log A - n \log R = \log 1000 - 20 \log 1 \cdot 05$$

$$\begin{array}{rr} \log 1 \cdot 05 = & \cdot 0211893 \\ & 20 \\ \hline & \cdot 423786 \\ \log 1000 = & 3 \\ \hline \log P = & 2 \cdot 576214 \\ \hline \end{array} \quad \therefore P = 376 \cdot 89 l. = 376 l.\ 17 s.\ 9\tfrac{1}{2} d.$$

3. In how many years will 210*l.* 6*s.* amount to 690*l.* at 8 per cent. compound interest payable quarterly?

$$P = 210 l.\ 6 s. = 210 \cdot 3 l.,\ A = 690 l.,\ r = \frac{\cdot 08}{4},\ \therefore R = 1 \cdot 02$$

By the formula, $n = \frac{\log A - \log P}{\log R}$

$$\begin{array}{rl} \log A = & 2 \cdot 8388491 \\ \log P = & 2 \cdot 3228393 \\ \hline \end{array}$$

$$\log R = \cdot 00860017\ )\ \cdot 5160098\ (\ 60 \text{ qrs.} = 15 \text{ years.}$$

$$\begin{array}{r} \cdot 5160102 \\ \hline \end{array}$$

4. In how many years will a sum of money P amount to m times itself, at compound interest, if r be the interest of £1 for a year?

$$A=mP=PR^n \therefore m=R^n \therefore \log m=n\log R \therefore n=\frac{\log m}{\log R}, \text{ where } R=1+r$$

Examples for Exercise.

(1) How much would £300 amount to in 4 years, at 4 per cent., compound interest?

(2) At what compound interest must £300 be placed out, so as to amount to £350. 19*s.* 2*d.* in 4 years?

(3) What will £600 amount to in 12 years, at 2½ per cent., compound interest?

(4) What is the compound interest on £275 for 3 years, at 5 per cent.?

(5) What sum will amount to £690 in 15 years, at 8 per cent., compound interest, payable quarterly?

(6) What is the present value of £350, due at the end of 10 years, allowing 5 per cent., compound interest?

(7) In what time will a sum of money double itself, at 5 per cent., compound interest?

(8) In what time will a sum of money double itself, at 3½ per cent., compound interest?

(9) At what rate of interest must £400 be placed out, so that it may amount to £569. 6*s.* 8*d.*, in 9 years, at compound interest?

(10) In what time will a sum of money treble itself, at 5 per cent., compound interest?

To find the amount of an annuity or pension in arrears.

68. Let U represent the annuity; if it be left unpaid n years, the amount of the first year's annuity (as one year elapses before it becomes due) will be UR^{n-1}, of the next year's annuity UR^{n-2}, and so on, while the last, or nth year's annuity, being paid in that year, that is, when due, is simply U. Hence the whole amount of the accumulated annuities is

$$A=U(R^{n-1}+R^{n-2}+R^{n-3}+\ldots\ldots+1)$$

The expression in the vinculum is a geometrical progression of n terms; therefore, by the formula for the sum of such progression, page 74, we have

$$A=U\frac{R^n-1}{r}$$

(1) What will an annuity of £30. 6*s.* 4*d.* amount to in 12 years, at 5 per cent., compound interest?

$$U=30l.\ 6s.\ 4d.=30\cdot31667l.,\ n=12,\ r=\cdot05 \therefore R=1\cdot05$$

$$\therefore A=30\cdot31667\frac{1\cdot05^{12}-1}{\cdot05}$$

$$\begin{array}{lr} \log 1{\cdot}05 = & {\cdot}0211893 \\ & 12 \\ \hline \log 1{\cdot}05^{12} = & {\cdot}2542716 \end{array} \quad \therefore \quad \begin{array}{r} 1{\cdot}05^{12} = 1{\cdot}7958563 \\ 1 \\ \hline {\cdot}7958563 \end{array}$$

$$\begin{array}{rr} 30{\cdot}31667 \text{ reversed } = & 7661303 \\ \hline & 23875689 \\ & 238757 \\ & 7959 \\ & 4775 \\ & 477 \\ & 55 \\ \hline {\cdot}05\,) & 24{\cdot}127712 \\ \hline & 482{\cdot}5542 \\ \hline \end{array}$$

The multiplication is here performed by the contracted method, which should always be used when the factors consist of several places of decimals. The amount is thus found to be $A=$ 482*l.* 11*s.* 1*d.*

Examples for Exercise.

(1) What will an annuity of £50 amount to in 4 years, at 5 per cent compound interest?

(2) What will an annuity of £75 amount to in 10 years, at 6 pe cent., compound interest?

(3) What will an annuity of £30 amount to in 16 years, at $4\frac{1}{2}$ pe cent., compound interest?

(4) What will a pension of £40 per annum amount to in 6 years, a 6 per cent., compound interest?

To find the present worth of an annuity or pension.

69. The present worth, P, is such a sum that if it were put out a the proposed rate of interest for the time during which the annuity i forborne, it would amount to the same sum as the annuity woul amount to. The amount of P in n years is PR^n,

$$\therefore\ PR^n = U\frac{R^n-1}{r} \quad \therefore\ \ P=\frac{U}{r}\left(1-\frac{1}{R^n}\right)$$

If the annuity be a *perpetual* annuity, or to continue for ever, the n being infinite, R^n will be infinite, and therefore $\frac{1}{R^n}=0$; hence

$$P=\frac{U}{r} \text{ is the present value of a perpetuity of } U,$$

which is, therefore, found by dividing the annuity by the interest of £ for a year. If U be the rent of a freehold estate, then, as the presen worth is $\frac{1}{r}U$, $\frac{1}{r}$ will express the number of years' purchase it is worth.

(1) What is the present value of an annuity of £170, to continue 20 years, at 5 per cent., compound interest?

$$U=170,\ n=20,\ r=\cdot05,\ \therefore\ R=1\cdot05$$

$$\log 1\cdot05^{-20}=-20\log 1\cdot05=-\cdot423786\ \therefore\ 1\cdot05^{-20}=\cdot37688948$$

$$\therefore\ 1-R^{-n}=\cdot62311052;\ \text{and}\ \frac{U}{r}=\frac{170}{\cdot5}=3400$$

$$\begin{array}{r} 3400 \\ \hline 249244208 \\ 186933156 \\ \hline 2118\cdot575768 \\ \hline \end{array}\ \therefore\ P=2118l.\ 11s.\ 6d.$$

(2) What is the present value of an estate in perpetuity of £250 per annum, allowing interest at 4 per cent.?

$$\frac{U}{r}=\frac{250}{\cdot04}=6250\ \therefore\ P=6250l.=25\ \text{years' purchase.}$$

And this is the number of years' purchase whatever be the rent; it is the same whether the rent be paid yearly or at any other equal periods, provided interest be regarded as payable at the same periods.

Examples for Exercise.

(1) What is the present value of an annuity of £1,000, to continue 3 years, at 4 per cent., compound interest?

(2) What is the present worth of an annuity of £30, to continue for 30 years, at 3½ per cent., compound interest?

(3) How many years' purchase must be paid for a freehold estate, when the interest of money is 3½ per cent.?

(4) A house is let for £35 a year; the tenant wishes to purchase a 7 years' lease for ready money: how much must he pay, if 6 per cent. be agreed upon for the interest of money?

(5) A person wishes to dispose of his salary of £40 a year for 8 years, allowing 5 per cent., compound interest: how much should he receive?

(6) What perpetual annuity will £925 purchase, allowing 5 per cent., compound interest?

(7) If £925 purchase a perpetuity of £46. 5*s*. per annum, what rate of interest is obtained?

To find the present worth of an annuity in reversion.

70. An annuity is said to be in reversion when the payment of it does not commence till a certain number of years have elapsed.

If the annuity continue n years, but be deferred d years, then the present value will be equal to that of the same annuity in possession for $n+d$ years diminished by the present value of it for d years.

$$\begin{aligned}\therefore\ P&=\frac{U}{r}\left(1-\frac{1}{R^{n+d}}\right)-\frac{U}{r}\left(1-\frac{1}{R^{d}}\right)\\ &=\frac{U}{r}\left(\frac{1}{R^{d}}-\frac{1}{R^{n+d}}\right)=\frac{1}{R^{d}}\cdot\frac{U}{r}\left(1-\frac{1}{R^{n}}\right)\end{aligned}$$

Consequently, the present value of the reversion is found by multiplying the present value of the annuity for n years by the present value of £1, receivable at the end of d years. When the reversion is in perpetuity, then n is infinite, and $\frac{1}{R^n}$ vanishes; the expression for the present value is, therefore, in this case,

$$P=\frac{U}{rR^d}$$

(1) What is the present value of the reversion of £30 per annum for 8 years, to be entered upon after the expiration of the next 10 years, compound interest, at 5 per cent.?

$$U=30,\ n=8,\ d=10,\ r=\cdot05\ \therefore\ R=1\cdot05$$

$$\begin{array}{rr} R^{-d} =1\cdot05^{-10}= & \cdot613913 \\ R^{-n-d}=1\cdot05^{-18}= & \cdot415521 \\ \hline & \cdot198392 \\ & 30 \\ \hline \cdot05\,) & 5\cdot95176 \\ \hline & 119\cdot035 \\ \hline \end{array}\quad \therefore\quad P=119l.\ 0s.\ 8\tfrac{1}{2}d.$$

(2) What is the present value of the reversion of a perpetuity of £50 per annum, after 10 years, at 5 per cent. interest?

$$U=50,\ d=10,\ r=\cdot05\ \therefore\ R=1\cdot05$$

$$\begin{array}{rr} R^{-d}=1\cdot05^{-10}= & \cdot613913 \\ & 50 \\ \hline \cdot05\,) & 30\cdot69565 \\ \hline & 613\cdot913 \\ \hline \end{array}\quad \therefore\quad P=613l.\ 18s.\ 3\tfrac{1}{4}d.$$

Examples for Exercise.

(1) What is the present value of the reversion of a lease of £40 per annum, to continue for 6 years, but not to commence till after 2 years, allowing 6 per cent. interest?

(2) What is the present value of a freehold estate, to commence 4 years hence, allowing interest at 5 per cent.?

(3) The reversion of a freehold estate, after 10 years, is sold for £613. 18s. 3¼d.: what annual rent should it produce to allow the purchaser 5 per cent. interest for his money?

[For additional examples on these subjects, see the last edition of Walkingame's Arithmetic (published by Routledge & Co.), also Jones's Annuities, &c. (published by direction of the Society for the Diffusion of Useful Knowledge). Some of the examples given above have been taken from these works. Jones's volumes contain an extensive collection of tables for abridging all calculations concerning Interest, Annuities, &c.]

On the increase in the population of a country.

71. If the population of a country at any time be represented by P, and the increase in one year be equal to a part of the whole represented by iP; then the amount of the population in the beginning of the second year, or end of the first, will be $P+iP=P(1+i)$; and, the increase being supposed to follow the same uniform law, the amount of population at the beginning of the third year, or end of the second, will be

$$P(1+i)+iP(1+i)=P(1+1)(1+i)=P(1+i)^2$$

at the end of the third, $P(1+i)^2+iP(1+i)^2=P(1+i)^3$

„ „ fourth, $P(1+i)^3+iP(1+i)^3=P(1+i)^4$

⋮

„ „ nth, $P(1+i)^{n-1}+iP(1+i)^{n-1}=P(1+i)^n$

The state of the population at the beginning of the first year of the enumeration being P, it appears, therefore, that its states at the end of the years 1, 2, 3, &c., will form the geometrical progression

$$P(1+i),\ P(1+i)^2,\ P(1+i)^3,\ldots\ldots P(1+i)^n$$

the common ratio being $1+i$. Calling this common ratio C, and the population at the end of the nth year P_n, we have

$$P_n=P\cdot C^n\ \therefore\ \log P_n=\log P+n\log C\ldots\ldots(\text{A})$$

The constant ratio C is

$$C=\left(\frac{P_n}{P}\right)^{\frac{1}{n}}=\frac{P_1}{P},\ \text{since } n \text{ is } \textit{any} \text{ integer.}$$

So that if a census of the population of a country were taken at the beginning and at the end of a year (or any other period), when there may be little or no reason to suspect that the law is sensibly disturbed by extraneous causes, such as emigration, war, famine, epidemic diseases, &c., and the resulting value of $\frac{P_1}{P}$ be taken, it may be compared with the value $\left(\frac{P_n}{P}\right)^{\frac{1}{n}}$ furnished by a census taken n years (or periods), from the aforesaid beginning; the difference between the two values will indicate the extent to which the natural law has been disturbed, during the interval, by the operation of extraneous causes.

If $P_n=mP$, then (A)

$$\log m+\log P=\log P+n\log C\ \therefore\ n=\frac{\log m}{\log C}\ldots\ldots(\text{B})$$

which expresses the number of years in which the population will have increased to m-fold.

The number iP is the part of the whole population by which it is increased in a year, so that $100i$ is the increase per cent.; i is, of course, the difference between the fraction expressing the births and that expressing the deaths.

72. In this kingdom a census of the population is taken every 10 years; suppose the population P in ten years to have increased p per cent., then at the end of the 10 years the amount will be

$$P+\frac{p\cdot P}{100}=P_{10}$$

By (A) $\quad P_{10}=P\cdot C^{10} \quad \therefore \quad P\cdot C^{10}=P\left(1+\frac{p}{100}\right)$

$$\therefore \quad C^{10}=1+\frac{p}{100} \quad \therefore \quad 10\log C=\log\left(1+\frac{p}{100}\right)=\log(100+p)-2$$

$$\therefore \quad \log C=\frac{1}{10}\{\log(100+p)-2\}$$

Substituting this in (B), we have

$$n=\frac{\log m}{\frac{1}{10}\{\log(100+p)-2\}}=\frac{10\log m}{\log(100+p)-2}\dots\text{(C)}$$

for the number of years in which the population will be increased to m-fold, supposing nothing to disturb the rate of increase observable during the preceding 10 years.

The population of England and Wales in 1851 was 17922768
" " " 1841 " 15911725

Increase in the 10 years 2011043

Hence the increase per cent was $\frac{201104300}{15911729}=12\cdot64$.

In the formula (C), suppose $m=2$, then, for the number of years in which the population of 1851 will be doubled, we have

$$n=\frac{10\log 2}{\log 112\cdot64-2}=\frac{3\cdot0103}{\cdot0517}=58 \text{ years}$$

But the census of 1851 was taken 68 days earlier than that of 1841, so that n, instead of beiug so many years, is, in fact, only about so many ·98 years; and $58\times\cdot98=56\cdot84$; so that, the rate of increase remaining the same, the population of 1851 may be expected to double itself in about 56 years and $\frac{5}{6}$.

[The preceding investigation is based upon what has been called the hypothesis of Euler; but it can scarcely be called an arbitrary hypothesis; if every individual in a population P, after the lapse of any time, count as $1+$the ith part, then it is a truism that—all circumstances remaining the same—every individual in the future population P_n would also count as $1+$the ith part, after the lapse of the same time. And this is all that is assumed above. Any deviation from uniformity should be attributed to change of circumstances. Occasional departures from the natural law—more or less wide—must, of course, be expected, from emigration, scarcity of food, &c.]

The principle of indeterminate coefficients.

73. The principle of indeterminate coefficients has already been applied in the investigations of the binomial and exponential theorems; its further application to the determination of the sums of series will be sufficiently understood from the following example.

(1) Required the sum of n terms of the series

$$1+2^2+3^2+4^2+\ldots\ldots+n^2$$

The sum will, of course, be some expression involving n; and if this expression be developed and orderly arranged according to the powers of n, it would necessarily be of the form $A+Bn+Cn^2+Dn^3+$&c. Put, therefore,

$$1+2^2+3^2+4^2+\ldots\ldots+n^2=A+Bn+Cn^2+Dn^3+\text{ \&c.}$$

then if the unknown or (as they are somewhat improperly called) the indeterminate coefficients A, B, C, &c., can be found, the expression for the sum will be obtained. If we change n into $n+1$, the preceding identity will be

$$1+2^2+3^2+4^2+\ldots\ldots+(n+1)^2=$$
$$A+B(n+1)+C(n+1)^2+D(n+1)^3+\text{\&c.}$$

so that subtracting the former identity from this, we shall have

$$n^2+2n+1=B+2Cn+C+3Dn^2+3Dn+D$$

the "&c." being suppressed because, as the two members are *identical*, and that no term containing n^3 or higher powers enters the first member, so can no term containing these powers enter the second member. Hence (47), equating the coefficients of the *like* powers of n, we have

$$\begin{array}{rcr} 3D=1 & \therefore & D=\frac{1}{3} \\ 2C+3D=2 & & C=\frac{1}{2} \\ B+C+\ D=1 & & B=\frac{1}{6} \end{array}$$

To find A. we have only to refer to the first identity; if in the first member of that we put $n=0$, the whole is 0, while the second, on the same assumption for n, becomes A, $\therefore A=0$; consequently,

$$1+2^2+3^2+\ldots\ldots+n^2=\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}=\frac{n+3n^2+2n^3}{6}$$
$$=\frac{2n^3+n^2+2n^2+n}{6}=\frac{n^2(2n+1)+n(2n+1)}{6}$$
$$=\frac{n(n+1)(2n+1)}{6}\ldots\ldots(\text{A})$$

(2) Required the sum of n terms of the series

$$1+3+6+10+15+21+28+\text{ \&c.}\ldots\ldots(1)$$

where the term in any place is equal to the *number* of terms in the series $1+2+3+4+5+$ &c. which denotes that place; thus, the second term 3 is the sum of two terms, the third term 6 the sum of three terms, and so on, the nth term being $\frac{n(n+1)}{2}$ (page 73).

The sum of the series (1) may be readily found from the result in the former example; thus, 1st. Suppose n to be an even number, then taking the terms of (and two, we shall have to sum $\frac{n}{2}$ terms of the series $4+16+36$ - that is, $\frac{n}{2}$ terms of $4(1+2^2+3^2+$ &c.) Putting, therefore, $\frac{n}{2}$ fo the expression (A), we have for the sum required

$$4\frac{\frac{1}{2}n\left(\frac{1}{2}n+1\right)(n+1)}{6}=\frac{n(n+1)(n+2)}{6}$$

2nd. Suppose n to be an odd number, then $n-1$ will be an evei ber; and by putting $n-1$ for n, in the expression for the sum even number of terms, we shall have

$$\frac{(n-1)(n)(n+1)}{6}$$

for the sum of all the terms but the last, or nth, which nth te seen above, is $\frac{n(n+1)}{2}$. Consequently the sum of n terms is

$$\frac{(n-1)n(n+1)}{6}+\frac{3n(n+1)}{6}=\frac{n(n+1)(n-1+3)}{6}=\frac{n(n+1)(n+2)}{6}$$

which is the same expression as before. Hence (B) is the su terms of the series, whether n be even or odd.

But without making any distinction of cases, the sum S may b expeditiously found as follows:—

$$S=1+3+6+10+\ldots\ldots+\frac{n(n+1)}{2}$$

$$S=\quad 1+3+6+\ldots\ldots+\frac{(n-1)n}{2}+\frac{n(n+1)}{2}$$

$$\therefore\ 2S=1+2^2+3^2+4^2+\ldots\ldots+n^2+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{6}=\frac{n(n+1)(2n+4)}{6}$$

$$\therefore\ S=\frac{n(n+1)(n+2)}{6}$$

Examples for Exercise.

It is required to find the sum of n terms of each of the fol series by the method of indeterminate coefficients.

(1) $1+2+3+4+\ldots\ldots\ldots+n$
(2) $1^2+3^2+5^2+\ldots\ldots\ldots+(2n-1)^2$
(3) $1^3+2^3+3^3+\ldots\ldots\ldots+n^3$
(4) $1\cdot2+2\cdot3+3\cdot4+\ldots\ldots+n(n+1)$

The answer to the first example is $\frac{n(n+1)}{2}$, and the answer third is $\left\{\frac{n(n+1)}{2}\right\}^2$, so that we have this remarkable property, namely, that

$$1^3+2^3+3^3+\ldots\ldots+n^3=(1+2+3+\ldots\ldots+n)^2$$

whatever integer n may be. This curious relationship between the sum of the cubes and the square of the sum of any number of terms of the series 1, 2, 3, &c., was first noticed by an old writer (Peletarius), and published by him in 1558 (See Powell's History of Natural Philosophy, p. 123).

To DETERMINE THE NUMBER OF SHOT IN A PILE.

74. Cannon shot and shells are deposited in arsenals in piles made up of horizontal courses or layers of shot, the base or bottom course being either an equilateral triangle, a square, or a rectangle; so that there are three kinds of pile—the triangular pile, the square pile, and the rectangular, or oblong pile, the denomination of the pile referring to the form of the base. Each side of a pile presents an inclined face, consisting of rows one above another, each row diminishing by a single shot. In the triangular and square piles, there is but one shot at top; in the rectangular pile, there is a single row of shot; the former two piles terminating in one shot, the latter pile in a ridge of shot.

Triangular Pile.

(1) Let it be required to find how many shot there are in a triangular pile, of which each side of the base contains 12 shot.

Each course is an equilateral triangle; the side of the bottom course is 12 shot, of the next 11 shot, of the next 10, and so on; hence, the number of shot in the bottom course will be the sum of an arithmetical progression, whose first term is 1 and last term 12; for the number in the next course the last term of the progression will be 11, for the number in the next the last term will be 10, and so on; we shall, therefore, find the entire number in the pile by first summing the progression $1+2+3+\ldots\ldots+12$, subtracting from this sum 12, from the remainder 11, from the remainder 10, and so on, till only a single shot is left, and then adding the aforesaid sum and these remainders together, thus:—

$1+2+3+\ldots\ldots+12=13\times6=78$ shot in the bottom course,

$\therefore\ 78+66+55+45+36+28+21+15+10+6+3+1=364$ shot in the whole pile.

But, instead of thus summing up the twelve courses, it is better to use the formula at page 112, for the sum of n terms of the triangular numbers, as the numbers 1, 3, 6, 10, &c., are called; thus, in the present case,

$$S=\frac{n(n+1)(n+2)}{6}=\frac{12\cdot13.14}{6}=2\cdot13\cdot14=364$$

Square Pile.

As the courses here are a series of squares, we have the number of shot expressed by the formula (A), page 111, namely,

$$S=\frac{n(n+1)(2n+1)}{6}$$

where n is the number of shot in a side of the base.

(2) Let it be required to find the number of shot in a square pile, of which a side of the base contains 12 shot.

$$S=\frac{n(n+1)(2n+1)}{6}=\frac{12\cdot13\cdot25}{6}=2\cdot13\cdot25=650$$

so that the number of shot in the pile is 650.

It is easy to see that the difference between the two piles, having the same number n of shot in a side of the base of each, is $\frac{n(n+1)(n-1)}{6}$; the difference of the two piles above is thus $\frac{12\cdot13\cdot11}{6}=2\cdot13\cdot11=286$.

If only one of the two preceding formulæ be remembered, you may find it an aid towards the recovery of the other if you only recollect that the last factor in one gives that of the other by transposing the coefficients in it; thus, $1n+2$ suggests in this way $2n+1$; as the square pile contains the greater number of shot for the same value of n, you can never be at a loss as to whether the factor $2n+1$ belongs to the one or to the other.

Rectangular Pile.

(3) Required the number of shot in a rectangular pile, of which the shorter side of the base contains 12 shot, and the longer side 16.

First imagine a square pile having 12 shot in a side of the base: to one of the slant faces imagine an equal face of shot to be applied; you will then have a rectangular pile, the sides of the base containing 12 and 13 shot respectively, and the upper ridge consisting of 2 shot. Let another face be added, and so on, till $16-12=4$ faces have been added, you will then have the proposed rectangular pile, the sides of the base containing 12 and 16 shot, respectively, and the upper ridge consisting of one shot more than the number of faces added, that is, of 5 shot. The mode of proceeding is thus obvious; we are first to find the number of shot in the square pile of base 12^2, and then to add to this sum 4 times one of its faces, that is, 4 times $1+2+3+\ldots\ldots+12$, or $4(13\cdot6)=312$, thus:

$$\frac{12\cdot13\cdot25}{6}=2\cdot13\cdot25=650, \text{ the square pile,}$$

$$4(1+2+3+\ldots\ldots+12)=312, \text{ the added faces,}$$

$$\text{the whole number of shot } \underline{\overline{962}}$$

It is plain from this, that if m be the number of shot in the longer side of the base, and n the number in the shorter side, the general formula for the number of shot in the rectangular pile will be

$$S=\frac{n(n+1)(2n+1)}{6}+(m-n)\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)(2n+1+3m-3n)}{6}$$

$$=\frac{n(n+1)(3m-n+1)}{6}$$

Thus, in the case above, where $m=16$ and $n=12$, we have

$$S=\frac{12\cdot 13\cdot 37}{6}=2\cdot 13\cdot 37=962$$

This form of the expression for S is the more easy to compute by; but the first form, being at once suggested by the nature of the case, is the more easily remembered.

As the number of shot in the top row is always $m-n+1$, if we call the top row t, then

$$3m=3t+3n-3, \therefore 3m-n+1=3t+2n-2,$$

and the formula for the sum is

$$S=\frac{n(n+1)(3t+2n+2)}{6}$$

Or if we count the top row, and omit 1 from n, the shorter side of the base, and put n' for $n-1$, the formula will be

$$S=\frac{n(n+1)\ (3t+2n')}{6}$$

Examples for Exercise.

(1) Required the number of shot in a triangular pile, of which each side of the base contains 8 shot.

(2) Required the number of shot in a triangular pile, of which each side of the base contains 30 shot.

(3) Required the number of shot in a square pile, of which each side of the base contains 7 shot.

(4) How many shot are there in a square pile, of which each side of the base contains 17 shot?

(5) How many shot are there in a rectangular pile, of which one side of the base contains 7 shot, and the other 16?

(6) How many shot are there in a rectangular pile, of which the shorter side of the base contains 30, and the top row 31 shot?

(7) Required the number of shot in an oblong pile, of which the number of courses is 15, and of which the top row contains 21 shot.

NOTE.—In any pile, the number of courses is equal to the number of shot in the shorter side, as is obvious.

ON PERMUTATIONS AND COMBINATIONS.

75. *Permutations.*—The different arrangements of which any set of things admits, as to the order in which they succeed one another, are

called the permutations of those things. Thus, the two letters *a*, *b*, admit of two such arrangements or permutations, namely, *ab* and *ba*; the letters *a*, *b*, *c*, admit of six permutations, namely, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*. Thus, the permutations of three things are $P_{(3)}=1\cdot2\cdot3$.

If there be four things *a*, *b*, *c*, *d*, then the fourth, *d*, may occupy any place among the three *a*, *b*, *c*; it may be 1st, 2nd, 3rd, or 4th, that is, may take either of four positions; and as *a*, *b*, *c* admit of 6 permutations, *a*, *b*, *c*, *d* must admit of 6 x 4 permutations; that is,

$$P_{(4)}=1\cdot2\cdot3\cdot4.$$

Similarly, if there be five things, *a*, *b*, *c*, *d*, *e*; then as *e* may take either of *five* positions among each of the permutations of *a*, *b*, *c*, *d*, it follows that $P_{(5)}=1\cdot2\cdot3\cdot4\cdot5$. And generally if there be *n* things,

$$P_{(n)}=1\cdot2\cdot3\cdot4\cdot5\ldots.n$$

But it is frequently required to find the permutations that a specified number out of a larger number of things admits of; as two things out of three, three out of seven, and so on; that is, to solve the following problem.

Problem 1. To find the number of permutations that can be formed with *m* things out of *n*.

Let *a*, *b*, *c*, *d*, &c. represent *n* things. And

1. Let $m=2$, so that the *n* things are taken in pairs; the number of these pairs in which *a* may stand first, will obviously be $n-1$; because *each* of the remaining $n-1$ things may be put after *a*. In like manner, the number of pairs in which *b* may stand first is also $n-1$; and the same as respects *c*, *d*, &c.; and as there are *n* things altogether, the whole number of permutations, when they are taken two and two, must be $n(n-1)$.

2. Let $m=3$; and suppose one of the *n* things, as *a*, to be taken away; then, if the remaining $n-1$ things be taken two and two, they will furnish $(n-1)(n-2)$ permutations, as just shown; and if the *a* taken away be now restored, and prefixed to each of these pairs in succession, we shall, of course, have $(n-1)(n-2)$ permutations of *threes*. Similarly, by suppressing *b*, *c*, *d*, &c. in succession, and proceeding in the same way, we shall have as many times $(n-1)(n-2)$ permutations of *threes*, as there are letters *a*, *b*, *c*, &c., that is, *n* times those permutations; hence the whole number of permutations that can be formed with *three* things out of *n* is $n(n-1)(n-2)$.

3. Let $m=4$; then removing one of the *n* things *a*, the permutations, three and three, of the remaining $n-1$ things, are $(n-1)(n-2)(n-3)$; and as any one of the *n* things may be thus removed, and then restored and prefixed to each permutation of threes in succession, it follows that the permutations of the *n* things, taken *four* at a time, will be $n(n-1)(n-2)(n-3)$. And generally, that the number of permutations of *n* things, taken *m* at a time, is

$$^{n}P_{m}=n(n-1)(n-2)(n-3)\ldots\ldots(n-m+1)$$

If $m=n$, then writing the factors in reverse order, the number is $1\cdot2\cdot3\cdot4\ldots.n$, as deduced before. But if $m=n-1$, we get the same product; for still writing the factors in reverse order, we have $2\cdot3\cdot4\ldots.n$. We conclude, therefore, that the number of permutations formed out of n quantities, by taking $n-1$ of them at a time, is the same as the number of permutations formed by taking them altogether.

When the n things are taken altogether, and it happens that, instead of being all different things, certain of them are repeated, then, of course, certain of the permutations must be repeated; to select those which are really different is the object of the next problem.

Problem 2. To find the number of *different* permutations of n things, taken altogether, when certain of them are repeated.

1. Suppose two only of the n things are alike: then as this pair enters *every* permutation, it enters $1\cdot2\cdot3\ldots\ldots n$ times; and as every permutation is accompanied by another in which the like things are simply interchanged, there can be only half of the entire number of permutations that are really different; to find these, therefore, we must divide the entire number by 2.

2. Suppose three of the n things are alike; then as many permutations as three things admit of, so many recurring sets of distinct permutations will the entire series of permutations consist of; and as we require to know the number in *one* distinct set only, it is plain that we must divide the entire number of permutations by $1\cdot2\cdot3$, the number of sets, which is the number of permutations due to the three like things. And, generally, if p of the n things are alike, there must be as many sets of the same permutations as there are permutations of these p things; hence the number of different permutations of n things, taken altogether, when p of them are alike, is

$$\frac{1\cdot2\cdot3\cdot4\ldots\ldots p\ldots\ldots n}{1\cdot2\cdot3\ldots\ldots p}=\frac{P_{(n)}}{P_{(p)}}=(p+1)(p+2)\ldots\ldots n$$

3. Suppose now that, besides the p like things, q others are alike; then the permutations just determined divide themselves into as many recurring sets as there are permutations of these q like things; so that, to get one of these sets, we must divide the foregoing expression by $1\cdot2\cdot3\ldots\ldots q$; hence the number of different permutations of n things taken altogether, of which p are alike, and also q others alike, is

$$\frac{1\cdot2\cdot3\cdot4\ldots\ldots n}{(1\cdot2\cdot3\ldots\ldots p)(1\cdot2\cdot3\ldots\ldots q)}=\frac{P_{(n)}}{P_{(p)}\times P_{(q)}}$$

and this reasoning may be extended to any number of sets of like things.

If we write the factors in the numerator in reverse order, after suppressing the p factors common to numerator and denominator, we have

$$\frac{P_{(n)}}{P_{(p)}P_{(q)}}=\frac{n(n-1)(n-2)\ldots\ldots(p+1)}{1\cdot2\cdot3\ldots\ldots q}$$

If $p+q=n \quad \therefore\ p=n-q$

$$\therefore\ \frac{P_{(n)}}{P_{(q)}P_{(n-q)}}=\frac{n(n-1)(n-2)\ldots\ldots(n-q+1)}{1\cdot2\cdot3\ldots\ldots\ldots q}$$

the number of permutations of n things of which q are alike, and also the remaining $n-q$ alike.

NOTE. The notation $P_{(n)}$ is used to imply that the n things are taken altogether, while ${}^nP_m$ indicates that they are taken only m at a time.

(1) How many changes can be rung with 8 bells?

$$P_{(8)}=1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8=40320$$

(2) How many changes can be rung with 5 bells out of 8?

$${}^8P_5=8\cdot7\cdot6\cdot5\cdot4=6720$$

(3) How many different permutations may be made of the letters in the word *Salamanca*, taking them altogether?

The number of letters here is 9, of which four are *a*'s and the others all different.

$$\therefore \frac{P_{(9)}}{P_{(4)}}=5\cdot6\cdot7\cdot8\cdot9=15120$$

(4) How many different permutations can be formed of the letters in the word *Mississippi*, taken altogether?

In these 11 letters 4 are *i*'s, 4 are *s*'s, and 2 are *p*'s.

$$\therefore \frac{P_{(11)}}{P_{(4)}P_{(4)}P_{(2)}}=\frac{5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11}{(1\cdot2\cdot3\cdot4)\,(1\cdot2)}=5\cdot7\cdot2\cdot9.5\cdot11=34650$$

Examples for Exercise.

(1) In how many different ways may 12 persons seat themselves at table?

(2) How many changes may be rung with 5 bells out of 6, and how many with the whole peal?

(3) How many different numbers may be formed with the figures in the number 211321313, taking them altogether?

(4) How many different numbers may be expressed by means of the figures 1, 2, 3, 4, taking them one at a time, two at a time, three at a time, and altogether, repetitions of the same figure being excluded?

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76. *Combinations.* Combinations are the different sets or parcels that can be formed out of a given number of things, under the restriction that in no two sets shall the things be all the same. Difference of arrangement merely, which is everything in permutation, is of no account in combinations; each combination must differ from every other by having in it at least *one* of the things which that other has not; thus, the combinations of *a, b, c, d,* taken three together, are *abc, abd, acd, bcd,* and no others; you see that, whichever of these four combinations you take, there is always at least one letter in it not to be found in either of the other three.

Problem 1. To find the number of combinations that can be formed out of n different things, when m of them are taken together.

1. Let the n things be taken two at a time; the number of *permutations* will be $n(n-1)$ as shown above. Each of these permutations, as *ab*, is accompanied by another, *ba*, so that where there are two permutations there is but one combination; hence, the number of combinations is $\frac{n(n-1)}{2}$

2. Let the n things be taken three at a time; the number of *permutations* will be $n(n-1)\,(n-2)$; but every simple combination, as

abc, furnishes $1\cdot2\cdot3$ of these permutations, as shown above; hence, the number of combinations is $\frac{n(n-1)(n-2)}{1\cdot2\cdot3}$. And, generally, when the n things are taken m at a time, the number of combinations is

$$^nC_m=\frac{n(n-1)(n-2)\ldots.(n-m+1)}{1\cdot2\cdot3\ldots.m}\ldots..(A)$$

This expression, as shown at page 117, is also that for the different *permutations* that can be formed out of n things taken altogether, when a set of them, m, are all alike, and the remaining set, $n-m$, are all alike; that is,

$$^nC_m=\frac{P_{(n)}}{P_{(m)}P_{(n-m)}}\ldots\ldots..(B)$$

If in (A) we put $m=1$, 2, 3, &c., successively, we have

$$^nC_1=n,\ ^nC_2=\frac{n(n-1)}{2},\ ^nC_3=\frac{n(n-1)(n-2)}{2\cdot3},\ \&c.$$

which are the several coefficients in the development of $(1+x)^n$; hence (page 89),

$$1+{}^nC_1+{}^nC_2+{}^nC_3+\ldots\ldots+{}^nC_n=2^n$$

so that the sum of the combinations that can be formed out of n things, by taking them first singly, then two at a time, then three, and so on, is 2^n-1.

In (B) put $n-m$ for m, then we have

$$^nC_{n-m}=\frac{P_{(n)}}{P_{(n-m)}P_{(m)}}\ldots..(C)$$

which is the same as the expression (B); consequently, the number of combinations of n things, taken $n-m$ together, is the same as the number of combinations of the same things taken m together, that is, $^nC_{n-m}={}^nC_m$.

Hence, whether ^nC_m or $^nC_{n-m}$ be computed by (A), the result is the same. The numerator and denominator of the expression for ^nC_m will each have m factors, the numerator and denominator for $^nC_{n-m}$ will each have $n-m$ factors; therefore, when ^nC_m is to be computed, it is better to change it for $^nC_{n-m}$, whenever $n-m$ is less than m, or whenever m exceeds $\frac{1}{2}n$.

(1) How many combinations may be formed of 9 things out of 12?

If we compute for $^{12}C_9$, there will be nine factors in each of the terms of the fraction (A); but if we compute for $^{12}C_3$, there will be only three: thus,

$$^{12}C_9={}^{12}C_3=\frac{12\cdot11\cdot10}{1\cdot2\cdot3}=220$$

(2) How many combinations may be formed of 5 things out of 12?

$$^{12}C_5=\frac{12\cdot11\cdot10\cdot9\cdot8}{1\cdot2\cdot3\cdot4\cdot5}=792$$

and this result would have been the same for $^{12}C_7$.

(3) How many combinations may be formed of 96 things out of 100?

$$^{100}C_{96}={}^{100}C_4=\frac{100\cdot 99\cdot 98\cdot 97}{1\cdot 2\cdot 3\cdot 4}=3921225$$

(4) Fifteen young ladies at a school walk out three a-breast for seven days in succession: it is required to arrange them daily so that no two shall walk a-breast twice.

The number of combinations that can be formed with 2 out of 15, is

$$^{15}C_2=\frac{15\cdot 14}{1\cdot 2}=105$$

Consequently, from the 15 young ladies, there may be formed 105 different couples. But from each three who walk side by side, 3 couples may be formed, and the restriction is that only one of these 3 couples is to be admitted in each rank; hence, only $\frac{1}{3}$ of the entire number of couples is admissible, $\therefore\ \frac{105}{3}=35$, the allowable number of sets of three, and as 5 such sets make up the whole school, they may take $\frac{35}{5}=7$ walks;* and may be arranged thus:—

1st day.	2nd day.	3rd day.	4th day.	5th day.	6th day.	7th day.
$a_1a_2a_3$	$a_1b_1c_1$	$a_1d_1e_1$	$a_1b_2d_2$	$a_1c_2e_2$	$a_1b_3e_3$	$a_1c_3d_3$
$b_1b_2b_3$	$a_2b_2c_2$	$a_2d_2e_2$	$a_2b_3d_3$	$a_2c_3e_3$	$a_2b_1e_1$	$a_2c_1d_1$
$c_1c_2c_3$	$a_3d_3e_3$	$a_3b_3c_3$	$a_3c_1e_1$	$a_3b_1d_1$	$a_3c_2d_2$	$a_3b_2e_2$
$d_1d_2d_3$	$b_3d_1e_2$	$d_3b_1c_2$	$b_1c_3e_2$	$c_1b_3d_2$	$b_2c_3d_1$	$c_2b_3e_1$
$e_1e_2e_3$	$c_3d_2e_1$	$e_3b_2c_1$	$d_1c_2e_3$	$e_1b_2d_3$	$e_2c_1d_3$	$d_2b_1e_3$

Examples for Exercise.

(1) How many combinations may be formed of 6 things out of 10?

(2) A person engages to entertain 15 friends as many evenings as they can come, 12 at a time, provided the same 12 do not come twice: how many evening parties must he receive?

(3) How many combinations may be formed out of 4 things, taken 1, 2, 3, and 4 at a time?

(4) How many combinations may be formed of 5 out of the 26 letters of the alphabet?

(5) How often may a different guard of 6 men be selected out of 60, under the restriction that no two are to mount guard together oftener than once?

77. It is sometimes required to find how often a single individual of of the n things can appear among the set of combinations; as, for instance, in Ex. 2, above, how often can any specified person appear at

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* The above question was proposed in the Lady's and Gentleman's Diary for 1850, by the Rev. T. P. Kirkman, rector of Croft, near Warrington, who exhibited the arrangement, as here given, in the Diary for 1851.

the parties? As there are 12 appearances each time, the total number of appearances is 455 × 12, and as each individual is $\frac{1}{15}$ of the whole, each must appear $\frac{1}{15}$ of 455 × 12 times, and similarly in other cases; so that, if n be the number of individuals, and m the number of selections from them, then ${}^{n}C_{m} \times \frac{m}{n}$ will be the number of times each will appear among the ${}^{n}C_{m}$ combinations. Or the matter may be considered otherwise, thus: suppose, instead of actual *visits*, in the above question, that the *cards* of the persons are left; then, at the end, the total number of cards will be 455 × 12, and as each of the 15 persons leaves the same number, each will have left $\frac{1}{15}$ of 455 × 12, that is 364.

[But to view the subject more generally; let it be required to find how often p of the n persons, visiting m at a time, will meet together. These p persons will evidently be combined with all the combinations of the remaining $n-p$ persons, out of the n persons, taken $m-p$ at a time. Now the number of combinations of $n-p$, taken $m-p$ at a time, is the same as the number of combinations taken $(n-p)-(m-p)$ at a time, by the principle at page 119, that is, taken $n-m$ at a time; so that we have either

$$ {}^{n-p}C_{m-p} = \frac{(n-p)(n-p-1)\ldots\text{ to } m-p \text{ factors}}{1\cdot2\cdot3\ldots\ldots\text{ to } m-p \text{ factors}} $$

$$ \text{or } {}^{n-p}C_{n-m} = \frac{(n-p)(n-p-1)\ldots\text{ to } n-m \text{ factors}}{1\cdot2\cdot3\ldots\ldots\text{ to } n-m \text{ factors}} $$

Each of these expresses how often the same p persons will be found together among the m persons selected out of the n persons. If $m-p$ be less than $n-m$, then, of course, it will be better to use the first of these formulæ, and, in the contrary case, to use the second. Let us take the case above, where $n=15$, and $m=12$, $\therefore\ n-m=3$.

$$ \begin{array}{lll} \text{If } p=\ 1; & \text{then } {}^{14}C_{3}=\dfrac{14\cdot13\cdot12}{1\cdot2\cdot3} & =364 \\ p=\ 2; & \quad,,\quad {}^{13}C_{3}=\dfrac{13\cdot12\cdot11}{1\cdot2\cdot3} & =286 \\ p=\ 3; & \quad,,\quad {}^{12}C_{3}=\dfrac{12\cdot11\cdot10}{1\cdot2\cdot3} & =220 \\ \vdots & \qquad\quad \vdots \qquad\quad \vdots & \ \ \vdots \\ p=12; & \quad,,\quad {}^{3}C_{3}=\dfrac{3\cdot2\cdot1}{1\cdot2\cdot3} & =\ 1 \end{array} $$

In this way may the desired result be found for any value of p, independently of any previously computed results. But if ${}^{n}C_{m}$ has to be determined, as in examples (3) and (5), we may avail ourselves of this value to shorten the subsequent work. For it is easy to see, from comparing the above general expressions with the expression (A) for ${}^{n}C_{m}$, that

$$ \left.\begin{array}{r} {}^{n-p}C_{m-p} \\ \text{or } {}^{n-p}C_{n-m} \end{array}\right\} = \frac{m(m-1)(m-2)\ldots\text{ to } p \text{ factors}}{n(n-1)(n-2)\ldots\text{ to } p \text{ factors}}\,{}^{n}C_{m} $$

a form which may very easily be remembered. Again, if ${}^{n-p}C_{m-p}$ or

${}^{n-p}C_{n-m}$ be calculated for any value of p, then the form for the next value, $p+1$, will be

$$ {}^{n-(p+1)}C_{m-(p+1)} = \frac{m-p}{n-p}\,{}^{n-p}C_{m-p}.] $$

(6) How often may a different guard of 6 men be taken out of 60: how often would the same man be among the 6: and how often would two comrades be found together?

By subtracting the last result from the first, in the answer to this question, we shall discover the number of times the guard may be selected, so that a particular couple of men may never be together.

(7) How often may the 15 young ladies in Ex. (4), page 120, walk out, under the restriction that a certain specified couple may never appear a-breast?

TO EXTRACT THE SQUARE ROOT OF A POLYNOMIAL.

78. The process for extracting the *square root* of a polynomial, that is, of an algebraical expression consisting of any number of terms, is suggested from an examination of the way in which the *square* of a polynomial is constructed, with a view to the discovery of the law of its terms.

If we first take a binomial $a+b$, then a trinomial $a+b+c$, then a quadrinomial $a+b+c+d$, and so on, to any polynomial $a+b+c+d+\ldots\ldots l$, we shall find the squares composed as follows:—

$$
\begin{aligned}
(a+b)^2 &= a^2+\{2a+b\}b \\
(a+b+c)^2 &= a^2+\{2a+b\}b+\{2(a+b)+c\}c \\
(a+b+c+d)^2 &= a^2+\{2a+b\}b+\{2(a+b)+c\}c+\{2(a+b+c)+d\}d \\
(a+b+c+d+e)^2 &= a^2+\{2a+b\}b+\{2(a+b)+c\}c+\{2(a+b+c)+d\}d \\
&\quad +\{2(a+b+c+d)+e\}e \\
\&c. &\qquad \&c.
\end{aligned}
$$

In these developments, the expressions $2a$, $2(a+b)$, $2(a+b+c)$, &c., are what, in the following rule, are called the *trial divisors*, and the complete expressions within the braces $\{\ \}$ are the corresponding complete divisors.

RULE 1. Having arranged the terms of the polynomial, as if for division, mark off to the right a place for the root.

2. In the place marked off, put the square root of the leading term of the polynomial, and the square of it under that term; draw a line, and bring down under it the next two terms of the polynomial; these are now to be regarded as a dividend, and a place is to be marked off to the left for a divisor.

3. Put twice the root-term just found in the divisor's place; you will thus have the first incomplete or *trial divisor*. Find how often it is contained in the leading term of the dividend, put the quotient with its proper sign in the root's place, and *also* connect it with the trial divisor; you will thus have the *complete divisor*.

4. Multiply the complete divisor by the last found root-term, sub-

tract the product from the dividend, and to the remainder annex the two next terms of the polynomial; you will thus have a second dividend.

5. Proceed now exactly as before; taking twice whatever is already in the root's place for a trial divisor, and from the leading term of this finding the next root-term, which, annexed to the trial divisor, renders it complete. And continue these steps till all the root-terms are obtained.

NOTE. If the polynomial be not a complete square, the process may be carried on to any extent.

That the above precepts are in accordance with what is shown above to be the composition of a square, is sufficiently seen from the operation below.

$$\begin{array}{lrl}
(1) & & a^2+\{2a+b\}b\,[a+b \\
 & & \underline{a^2} \\
 & 2a+b] & \{2a+b\}b \\
 & & \underline{\{2a+b\}b}
\end{array}$$

$$\begin{array}{lrl}
(2) & & a^2+\{2a+b\}b+\{2(a+b)+c\}c\,[a+b+c \\
 & & \underline{a^2} \\
 & 2a+b] & \{2a+b\}b \\
 & & \underline{\{2a+b\}b} \\
 & 2(a+b)+c] & \qquad\{2(a+b)+c\}c \\
 & & \qquad\underline{\{2(a+b)+c\}c}
\end{array}$$

$$\begin{array}{lrl}
(3) & & a^2+\{2a+b\}b+\{2(a+b)+c\}c+\{2(a+b+c)+d\}d\,[a+b+c+d \\
 & & \underline{a^2} \\
 & 2a+b] & \{2a+b\}b \\
 & & \underline{\{2a+b\}b} \\
 & 2(a+b)+c] & \qquad\{2(a+b)+c\}c \\
 & & \qquad\underline{\{2(a+b)+c\}c} \\
 & 2(a+b+c)+d] & \qquad\qquad\{2(a+b+c)+d\}d \\
 & & \qquad\qquad\underline{\{2(a+b+c)+d\}d}
\end{array}$$

The braces are preserved here for greater distinctness in showing the dependence of the several steps of the operation; if they be removed, each dividend will appear as *two* terms brought down from the polynomial, agreeably to the directions in the rule.

(4) Extract the square root of $4x^4-12x^3y+25x^2y^2-24xy^3+16y^4$

$$\begin{array}{ll}
 & 4x^4-12x^3y+25x^2y^2-24xy^3+16y^4\,[2x^2-3xy+4y^2 \\
 & 4x^4 \\
 & \overline{} \\
4x^2-3xy] & \quad -12x^3y+25x^2y^2 \\
 & \quad -12x^3y+9x^2y^2 \\
 & \quad \overline{} \\
4x^2-6xy+4y^2] & \qquad\qquad 16x^2y^2-24xy^3+16y^4 \\
 & \qquad\qquad 16x^2y^2-24xy^3+16y^4 \\
 & \qquad\qquad \overline{}
\end{array}$$

The proposed polynomial is formed from the terms of the root like (2) above; that is, it is

$$(2x^2)^2+\{4x^2-3xy\}(-3xy)+\{2(2x^2-3xy)+4y^2\}4y^2$$

and the expressions within the braces are the two complete divisors.

Extract the square root of $1+x$.

$$\begin{array}{ll}
 & 1+x\,[1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\text{\&c.} \\
 & 1 \\
 & \overline{} \\
2+\frac{x}{2}] & x \\
 & x+\frac{x^2}{4} \\
 & \overline{\phantom{x+\frac{x^2}{4}}} \\
2+x-\frac{x^2}{8}] & -\frac{x^2}{4} \\
 & -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
 & \overline{\phantom{-\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64}}} \\
2+x-\frac{x^2}{4}] & \qquad\frac{x^3}{8}-\frac{x^4}{64} \\
 & \qquad\text{\&c.}\quad\text{\&c.}
\end{array}$$

$$\therefore\ \sqrt{(1+x)}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{x^4}{32}+\text{\&c.}$$

the same development as furnished by the binomial theorem.

Examples for Exercise.

Extract the square root of

(1) $4x^4-24ax^3+0a^2x^2+108a^3x+81a^4$.

(2) $a^4x^2-12a^3x^3+28a^2x^4+48ax^5+16x^6$.

(3) $x^4-4x^3+10x^2+0x+9$.

(4) $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$.

The above operation for the square root sufficiently explains the arithmetical process for the square root of a number; the *pointing*, or placing a dot over every alternate figure—commencing at the units' figure—serves to show how many figures the root is to consist of; the bringing down a *period* at every step is analogous to the bringing down of two terms in the algebraic process.

TO EXTRACT THE CUBE ROOT OF A POLYNOMIAL.

79. The construction of a cube is as follows:—

$$(a+b)^3=a^3+\left\{3a^2+3ab+b^2\right\}b$$

$$(a+b+c)^3=a^3+\left\{3a^3+3ab+b^2\right\}b+\left\{3(a+b)^2+3(a+b)c+c^2\right\}c$$

$$\&c. \qquad\qquad\qquad \&c.$$

And this construction suggests the following rule for evolving the root from the cube; the expressions $3a^2$, $3(a+b)^2$, $3(a+b+c)^2$, &c., being the several trial divisors, and the complete expression within the braces the corresponding true or complete divisors.

RULE 1. Arrange the terms of the polynomial as in the square root, and put the cube root of the leading term in the root's place, and the cube of it under the leading term; draw a line underneath, and bring down three terms for a dividend. Three times the square of the root-term will be its trial divisor.

2. With the trial divisor and first term of the dividend find the second term of the root; annex to the trial divisor three times the product of the last found term and the preceding, as also the square of this last found term; the divisor will then be completed.

3. Multiply the complete divisor by the last found term, subtract the product from the dividend, and bring down three new terms.

4. Proceed now exactly as before, taking three times the square of all that is in the root's place for a trial divisor, and find the next term of the root; annex to the trial divisor three times the product of the last found term, and all that precedes it, and also the square of this last found term; the second divisor will thus be completed; and in this uniform way is the work to be carried on till the terms of the root are all obtained. If the polynomial have no cube root, the operation may, of course, be carried on indefinitely.

(1)

$$\begin{array}{lll}
 & a^3+\left\{3a^2+3ab+b^2\right\}b+\left\{3(a+b)^2+3(a+b)c+c^2\right\}c & [a+b+c \\
 & a^3 & \\
\cline{2-2}
3a^2+3ab+b^2] & \left\{3a^2+3ab+b^2\right\}b & \\
 & \left\{3a^2+3ab+b^2\right\}b & \\
\cline{2-2}
3(a+b)^2+3(a+b)c+c^2] & \qquad\left\{3(a+b)^2+3(a+b)c+c^2\right\}c & \\
 & \qquad\left\{3(a+b)^2+3(a+b)c+c^2\right\}c & \\
\cline{2-2}
\end{array}$$

If the braces be removed, three terms will have been brought down at each step, agreeably to the rule.

(2) Extract the cube root of $x^6+6x^5-40x^3+96x-64$.

$$\begin{array}{rl}
 & x^6+6x^5+0x^4-40x^3+0x^2+96x-64\;[x^2+2x\cdot \\
 & \underline{x^6} \\
3x^4+6x^3+4x^2] & 6x^5+0x^4-40x^3 \\
 & \underline{6x^5+12x^4+8x^3} \\
3(x^2+2x)^2-12(x^2+2x)+16\,\rceil & -12x^4-48x^3+0x^2+96x-64 \\
\text{or } 3x^4+12x^3+0x^2-24x+16\,\rfloor & \underline{-12x^4-48x^3+0x^2+96x-64}
\end{array}$$

Examples for Exercise.

Extract the cube root of

(1) $8x^3+12ax^2+6a^2x+a^3$.

(2) $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

(3) $64x^6-288x^5+1080x^3-1458x-729$.

From the foregoing operation, it is easy to see the foundation of rule for the cube root of a number, given in most books on arithm

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MISCELLANEOUS PARTICULARS AND INVESTIGATIONS.

80. Before concluding this compendium of algebra, it may be wel direct your attention to a few points of practical importance in refer to the forms in which the *results* of certain algebraical investigat should be given.

1. Radical signs should never be left in the *denominators* of fracti results, whenever there are means of removing them, especially w such results are to be actually computed in numbers; thus, such a re as $x=\frac{1}{\sqrt{7}}$ should be changed into $x=\frac{\sqrt{7}}{7}$, by multiplying numer and denominator of the former by $\sqrt{7}$, for this reason: the comp tion of $\frac{1}{\sqrt{7}}$ requires, after $\sqrt{7}$ is found, a long division operati whereas $\frac{\sqrt{7}}{7}$ requires only the simple division of $\sqrt{7}$ by 7; and time and trouble are saved to the computor. It is plain that $\frac{a}{\sqrt{\ }}$ always reducible to the usually more convenient form of $\frac{a\sqrt{b}}{b}$, by tiplying both terms of the former fraction by the irrational den nator; this change has already been recommended at page 38.

2. If the denominator consist of a *pair* of quadratic radicals, instance, as in the result $\frac{m}{\sqrt{a}\pm\sqrt{b}}$, the denominator may be rat alized by multiplying the terms of the fraction by $\sqrt{a}\mp\sqrt{b}$; since sum multiplied by the difference gives the differences of the squares

that $\dfrac{m}{\sqrt{a}\pm\sqrt{b}}=m\dfrac{\sqrt{a}\mp\sqrt{b}}{a-b}=\dfrac{m}{a-b}(\sqrt{a}\mp\sqrt{b})$; which last form is usually more easy of computation when the letters are replaced by numbers.

3. If the denominator consist of a pair of *cubic* radicals, it may still be rationalized; thus,

$$\frac{m}{\sqrt[3]{a}\pm\sqrt[3]{b}}=\frac{m}{\sqrt[3]{a}\pm\sqrt[3]{b}}\cdot\frac{\sqrt[3]{a^2}\mp\sqrt[3]{ab}+\sqrt[3]{b^2}}{\sqrt[3]{a^2}\mp\sqrt[3]{ab}+\sqrt[3]{b^2}}=\frac{m}{a\pm b}(\sqrt[3]{a^2}\mp\sqrt[3]{ab}+\sqrt[3]{b^2})$$

We shall give an example of each of the last two cases.

(1) $\dfrac{2+\sqrt{3}}{5+\sqrt{3}}=\dfrac{(2+\sqrt{3})(5-\sqrt{3})}{25-3}=\dfrac{10+3\sqrt{3}-3}{22}=\dfrac{7+3\sqrt{3}}{22}$

(2) $\dfrac{2}{\sqrt[3]{5}-\sqrt[3]{2}}=\dfrac{2(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})}{3}=\dfrac{2}{3}(\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4})$

4. When the form $\sqrt{(a\pm\sqrt{b})}$ occurs, the operation indicated by the outer radical may often be performed, or rather, the above complex surd may often be reduced to the sum or difference of two simple surds, that is, to the form $\sqrt{p}\pm\sqrt{q}$, as follows:—

81. *To extract the square root of a binomial, one of whose terms is rational, and the other a quadratic surd.*

Put $a+\sqrt{b}=(x+y)^2$ and $a-\sqrt{b}=(x-y)^2$

Adding, $\quad 2a=2x^2+2y^2 \quad \therefore \quad a=x^2+y^2 \ldots\ldots(1)$

Also $(a+\sqrt{b})(a-\sqrt{b})=(x^2-y^2)^2$, that is, $\sqrt{(a^2-b)}=x^2-y^2 \ldots\ldots(2)$

From (1) and (2), by adding and subtracting,

$$x^2=\frac{a+\sqrt{(a^2-b)}}{2},\quad y^2=\frac{a-\sqrt{(a^2-b)}}{2}$$

$$\therefore\ x+y=\sqrt{(a+\sqrt{b})}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}+\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}\ldots\ldots(3)$$

$$x-y=\sqrt{(a-\sqrt{b})}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}-\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}\ldots\ldots(4)$$

Consequently, if the numbers a, b, be such as to make a^2-b, a square, $=c^2$, we shall have

$$\sqrt{(a+\sqrt{b})}=\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}},\text{ and}$$

$$\sqrt{(a-\sqrt{b})}=\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}$$

In this case, therefore, the complex surd $\sqrt{(a\pm\sqrt{b})}$ can be reduced to the simpler form $\sqrt{p}\pm\sqrt{q}$, but not under other circumstances.

(1) Reduce $\sqrt{(16-\sqrt{87})}$ to a simpler form.

Here $16^2-87=169=13^2$, therefore

$$\sqrt{(16-\sqrt{87})}=\sqrt{\frac{16+13}{2}}-\sqrt{\frac{16-13}{2}}=\frac{1}{2}(\sqrt{58}-\sqrt{6})$$

(2) $\sqrt{(5+\sqrt{9})}=\sqrt{\frac{5+4}{2}}+\sqrt{\frac{5-4}{2}}=\frac{1}{2}(3\sqrt{2}+\sqrt{2})=2\sqrt{2}$

as we otherwise know, since $\sqrt{(5+\sqrt{9})}=\sqrt{(5+3)}=\sqrt{8}=2\sqrt{2}$.

These two examples have been worked by a reference to the formul (3), (4), but as these are less easily recollected than the process whic leads to them, it will be better to go through that process in eac particular case, so that the work of Ex. 1 would stand thus :

$$\text{Put } 16-\sqrt{87}=(x-y)^2 \ldots\ldots (1)$$

$$\text{and } 16+\sqrt{87}=(x+y)^2 \ldots\ldots (2)$$

$$\therefore \text{ multiplying, } 256-87=(x^2-y^2)^2 \quad \therefore \quad 13=x^2-y^2 \ldots\ldots (3)$$

$$\text{Adding (1) and (2), } \qquad 16=x^2+y^2$$

$$\therefore \text{ by adding and subtracting, } \quad 29=2x^2, \text{ and } 3=2y^2$$

$$\therefore x=\sqrt{\frac{29}{2}}=\frac{1}{2}\sqrt{58}, \text{ and } y=\sqrt{\frac{3}{2}}=\frac{1}{2}\sqrt{6}$$

$$\therefore \sqrt{(16-\sqrt{87})}=x-y=\frac{1}{2}(\sqrt{58}-\sqrt{6})$$

We should have stopped the operation at the step (3), if the number o the left-hand side of the equation had not been a square ; because w should then have known that the square root of $16-\sqrt{87}$ could not b converted into a form less complex than the form $\sqrt{(16-\sqrt{87})}$.* Th the form $\frac{1}{2}(\sqrt{58}-\sqrt{6})$ *is* less complex, you will at once perceive upo roughly estimating the amount of numerical work implied in each form to compute the square root from the original form, you would have fir to take the square root of 87 ; this, of course, you might get out *tables* of square roots, to six or seven places of decimals ; but when th root is subtracted from the 16, you would then have, secondly, to extract the square root of the remainder—a number consisting of to many figures to be found within the limits of the table ; here, the would be actual numerical work. But to compute the form $\frac{1}{2}(\sqrt{58}-\sqrt{6})$, we should merely have to take $\sqrt{58}$ and $\sqrt{6}$ out of the table and to divide their difference by 2. The most useful tables, of th kind here adverted to, are those of Mr. Barlow, reprinted by directio of the Society for the Diffusion of Useful Knowledge, as revised a corrected by Professor De Morgan.

82. The following properties of quadratic surds are both interesti and useful.

1. The square root of a quantity cannot be equal to a ration quantity plus or minus a quadratic surd, that is, $\sqrt{a}$ and $\sqrt{c}$, bei surds, we can never have the equality $\sqrt{a}=b\pm\sqrt{c}$; for, if we coul then, by squaring, we should have $a=b^2\pm 2b\sqrt{c}+c$, and, consequentl

---

* In $4+\sqrt{18}$, where $a^2-b=-2$, if we multiply by $\sqrt{2}$, we chan the expression into $\frac{6+4\sqrt{2}}{\sqrt{2}}$, the square root of which is $(2+\sqrt{2})+\sqrt[4]{}$ $=\sqrt[4]{8}+\sqrt[4]{2}$, a binomial surd, though not of the form $\sqrt{p}+\sqrt{q}$.

$\sqrt{c}=\pm\frac{a-c-b^2}{2b}$, that is, $\sqrt{c}=$ a rational quantity, which contradicts the hypothesis.

2. In every equation of the form $a+\sqrt{b}=x+\sqrt{y}$, if $\sqrt{b}$ and $\sqrt{y}$ be surds, then must $a=x$, and $\therefore b=y$; for, $\sqrt{b}=(x-a)+\sqrt{y}$, $\therefore$ by last principle $x-a=0$, otherwise $\sqrt{b}$ would be $=$ a rational quantity and a surd.

3. If $\sqrt{(a+\sqrt{b})}=x+y$, where x and y are one or both quadratic surds, then must $\sqrt{(a-\sqrt{b})}=x-y$, if $\sqrt{b}$ be a surd; for since $a+\sqrt{b}=x^2+2xy+y^2$, x^2+y^2 must be a rational quantity, and $2xy$ irrational, otherwise, upon transposing a, we should have $\sqrt{b}=$ a rational quantity,

$$\therefore (2),\ a=x^2+y^2 \text{ and } \sqrt{b}=2xy$$
$$\therefore\ a-\sqrt{b}=x^2+y^2-2xy=(x-y)^2\ \therefore\ \sqrt{(a-\sqrt{b})}=x-y$$

[It thus appears that if $\sqrt{(a+\sqrt{b})}$ is really convertible into the form $\sqrt{p}+\sqrt{q}$, then, of necessity, $\sqrt{(a-\sqrt{b})}=\sqrt{p}-\sqrt{q}$, when $\sqrt{b}$ is a surd, and one, at least, of the two $\sqrt{p}$, $\sqrt{q}$, a surd. There was, therefore, strict propriety, after having put $\sqrt{(a+\sqrt{b})}=x+y$ in article (81), in then putting $\sqrt{(a-\sqrt{b})}=x-y$; but if we had made the assumptions for *this* reason, as is usually done in the books, we should have restricted the results (3), (4) to the cases in which $\sqrt{b}$ is a surd; whereas, it is plain that the equations (3), (4) are *identities*, free from every restriction whatever; we have only to square each side to assure ourselves of this. The solution to Ex. 2 could not have been trusted without verification, if the investigation of the formulæ (3) and (4) had been founded on the present theorem, since $\sqrt{9}$ is *not* a surd. Having made the assumption $a+\sqrt{b}=(x+y)^2$, in Art. (81), it was, of course, allowable to make $a-\sqrt{b}=(x-y)^2$, or $=$ any other expression involving only the unknowns x and y, provided the equations, thus assumed, were found to admit of solution by rules previously established; but the form $a-\sqrt{b}=(x-y)^2$ was chosen rather than any other, with a prospective view to the desired form for the result.

It is true that the identities (3) and (4) are required, for arithmetical purposes, only for the cases in which $\sqrt{b}$ is a surd, but it is wrong to abridge the two members of each indentity of their complete algebraical equivalence under all circumstances, and, *consequently*, of their complete arithmetical equivalence when *any* numbers are put for the symbols, though, in common with other writers, the author has himself elsewhere committed the oversight.]

To find the greatest common measure of two quantities.

83. Let $A\gamma$, $B\gamma$ be two quantities, having the factor γ common to both, but no additional factor, so that A and B have no common measure; then $A\gamma$ being supposed to be the greater of the two quantities, let the successive divisions be carried on as below, till the operation terminates for want of a new remainder. Let d be the last of the quotients; then, since quotient multiplied by divisor, plus remainder, $=$ dividend, we have the following series of equalities, on the right of the operation.

$$\begin{array}{llllll}
B\gamma\,)\,A\gamma\,(\,a & & & & & \\
\quad aB\gamma & & & & & \\
\hline
\quad C\gamma\,)\,B\gamma\,(\,b & & & & & \therefore\ dE=D \\
\qquad bC\gamma & & & & & cD+E=C \\
\hline
\qquad D\gamma\,)\,C\gamma\,(\,c & & & & & bC+D=B \\
\qquad\quad cD\gamma & & & & & aB+C=A \\
\hline
\qquad\quad E\gamma\,)\,D\gamma\,(\,d & & & & & \\
\qquad\qquad D\gamma & & & & & \\
\hline
\end{array}$$

From the equations on the right, it is plain that E is necessarily a factor of A and B; but A and B, by hypothesis, have no common measure, therefore, $E=1$. It follows, therefore, that the final divisor $E\gamma$ is simply γ, the greatest common measure of the two proposed quantities. If the final divisor should be $\gamma=1$, we should then conclude that the quantities have no common measure.

A factor found to enter any divisor may be expunged, provided it do not also enter the dividend; or a factor may be expunged from a dividend, provided it do not enter the divisor; and, in like manner, into either a factor may at any time be introduced; for by these modifications no *common measure* is either suppressed or introduced; a glance at the general type of the operation is sufficient to show that the G. C. M. of $A\gamma$, $B\gamma$ is also the G. C. M. of $B\gamma$, $C\gamma$, of $C\gamma$, $D\gamma$, &c.

(1) Find the G. C. M. of $x^4-4x^3+6x^2-4x+1$ and x^4-2x^2+1

$$x^4-2x^2+1\,)\,x^4-4x^3+6x^2-4x+1\,(\,1$$
$$x^4-2x^2+1$$
$$\overline{-4x^3+8x^2-4x}$$

Expunging the factor $-4x$, $x^2-2x+1\,)\,x^4-2x^2+1\,(\,x^2+2x+1$

$$x^4-2x^3+x^2$$
$$\overline{2x^3-3x^2+1}$$
$$2x^3-4x^2+2x$$
$$\therefore\ x^2-2x+1=\text{G. C. M.} \qquad \overline{x^2-2x+1}$$
$$x^2-2x+1$$

(2) Find the G. C. M. of $12x^3+4x^2-3x+1$ and $8x^3-4x^2-2x+1$

Multiply the first expression by 2 to preclude a fraction in the quotient.

$$8x^3-4x^2-2x+1\,)\,24x^3+\ 8x^2-6x-2\,(\,3$$
$$24x^3-12x^2-6x+3$$
$$\overline{20x^2-5}$$

or, $4x^2-1\,)\,8x^3-4x^2-2x+1\,(\,2x-1$

$$8x^3-2x$$
$$\therefore\ 4x^2-1=\text{G. C. M.} \qquad \overline{-4x^2+1}$$
$$-4x^2+1$$

Examples for Exercise.

(1) Find the G. C. M. of a^4-x^4 and $a^3+a^2x-ax^2-x^3$.

(2) $9x^3+53x^2-9x-18$ and $x^2+11x+30$.

(3) $4x^3-25x^2+20x+25$ and $x^2-7x+10$.

(4) $2x^5-8x^4+12x^3-8x^2+2x$ and $3x^5-6x^3+3x$.

NOTE. If we divide the product of two quantities by their G. C. M., we shall have the least common multiple of the two quantities; thus, $\frac{A\gamma \times B\gamma}{\gamma}=AB\gamma$, the least quantity which is divisible by both $A\gamma$ and $B\gamma$, as is obvious, since A and B having no factor in common, their product must be the least quantity containing both.

The foregoing method of investigating the theory of the G. C. M. is more short and simple than that given in the books. It is useful in reducing fractions to their lowest terms; but, on account of the frequent length of the operation, it is resorted to only in cases of necessity, where it is desirable to simplify the fraction to the utmost, and where the composition of numerator and denominator is not discoverable by inspection. The process, however, is worth attention for other purposes; it is closely connected with important inquiries respecting the solution of the higher equations. (See the author's Analysis and Solution of Cubic and Biquadratic Equations.)

Before closing the present subject, it may be well to direct the attention of the student to a peculiarity respecting equations involving irrational terms, which is well deserving of notice, but which is likely to escape observation unless specially pointed out.

At page 58, the equation $2x-3\sqrt{x}=9$, involving the irrational term $3\sqrt{x}$, is proposed for solution, and it is remarked that instead of removing the irrational term in the usual way, "it is better to put y for $\sqrt{x}$, and thus to change the equation into $2y^2-3y=9$."

In order to show *why* this is better, let us clear the radical in the ordinary manner. The solution of the equation will then stand thus:—

$$3\sqrt{x}=2x-9 \quad \therefore \quad 9x=4x^2-36x+81$$

$$\therefore \ 4x^2-45x=-81$$

$$\therefore \ \text{(Rule II.)} \qquad 8x-45=\sqrt{729}=\pm 27$$

$$\therefore \ x=\frac{45\pm 27}{8}=9 \text{ or } \frac{9}{4}$$

These values of x are, of course, the same as those obtained at page 58; but the advantage of the former solution is this: we learn from it not only what each value of x is, but, moreover, what each value of $\sqrt{x}$ is,—an important peculiarity, since, for the verification of the proposed equation, the value of $\sqrt{x}$ as well as of x must be explicitly stated. In the solution just given we cannot tell whether, for $x=9$, $\sqrt{x}$ is 3, or -3; nor, for $x=\frac{9}{4}$, whether $\sqrt{x}$ is $\frac{3}{2}$, or $-\frac{3}{2}$. But the solution at page 58 expressly declares that y, or $\sqrt{x}$, must be in the one case 3 exclusively, and in the other case $-\frac{3}{2}$ exclus If $\frac{9}{4}$ were put for x and $\frac{3}{2}$ for $\sqrt{x}$, in the proposed equation, the in condition would be contradicted, for instead of 9, the first me would give

$$\frac{9}{2} - \frac{9}{2} = 0.$$

The most explicit form, therefore, of writing the values of x v be $x=3^2$, or $x=\left(-\frac{3}{2}\right)^2$; and it appears from the present illustı that whenever the unknown quantity occurs under a radical, i equation proposed for solution, caution must be exercised in veri the numerical result of that solution, as to which of the ambi signs ± belongs exclusively to the *root* of that result.

It will be observed that at page 58 the radical $\sqrt{x}$ is virtuall served throughout the entire process;—it is merely replaced by an symbol, y; but in the operation above, it has been removed at the step, and thus its peculiarity as to sign is no longer impressed upo work; so that at the close, we are unable to say, without actual whether the proposed condition requires that x be generated fron one of its two roots or from the other. Whenever radicals are in way removed from an equation, it is only by actual substitution c results in the original equation that the proper sign, which the n rical root in each case should take, can be discovered. But for a ample examination of matters of this kind, the student is referred t Algebra and Key, by the author of this Compendium, publish Mozley and Co., Paternoster Row.

We here terminate the Elementary Algebra of the present Co The student will bear in mind that we call it only a *Compendium*. its fullest extent the subject has a very comprehensive range; and is here given should be regarded only as a sufficient preparation fo study of algebraical writings of wider scope, and of higher preten

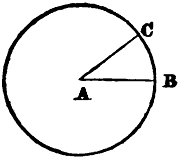
END OF THE ALGEBRA.

II.—PLANE TRIGONOMETRY.

PART I.

84. **Trigonometry** strictly means the measurement of *triangles*, or three-angled figures; but in its modern acceptation the term includes more than this, and is applied to the science which investigates the general theory of angular magnitude, taking in the angles of a triangle only as a particular branch of the subject. It is divided into two parts, plane and spherical, according as we consider the angles to be formed upon a plane, or upon a spherical surface; our business at present is with plane trigonometry only.

On the measurement of Angular Magnitude.



Let BAC be an angle; and with centre A and any radius AB, let a circle be described, then by Euclid, Prop. 33, Book vi., the arc varies as the angle which it subtends; that is, the arc BC is the same part of the entire circumference, that the angle A is of four right angles, or of the sum of all the angles that can be placed, side by side, about the centre A.

The circumference of the circle is conceived to be divided into 360 equal parts, called *degrees;* the degree into 60 equal parts, called *minutes;* and the minute into 60 equal parts, called *seconds;* the still smaller parts are represented as fractions or decimals of the second.

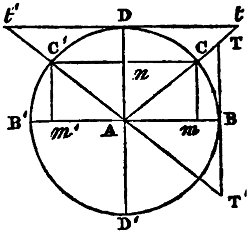
Degrees, minutes, and seconds are represented by a particular notation; thus, $37^\circ\ 14'\ 23''$ denotes 37 degrees, 14 minutes, and 23 seconds; and if this be the number of degrees, &c., in the arc BC, it is said to be *also* the number of degrees, &c., in the angle A, which it subtends. The angle is thus said to *measure* so many degrees, minutes, &c., with the understanding that 4 right angles measure 360°; and there is just the same propriety in this, as in saying that a certain length measures $3\frac{1}{4}$ inches, with the understanding that a foot measures 12 inches.

The thing called a *degree*, remember, is the 360th part of a *whole*, either of the whole circumference, or of the whole of the angular magnitude about a point—4 right angles; so that the expressions "arc of 15°" and "angle of 15°" are equally intelligible; the one means the 24th part of a circumference, the other the 24th part of 4 right angles. As an angle is thus expressed by the same number of degrees and parts of a degree as the arc which subtends it, it is usual to say that the arc *measures* the angle; *measure* is not to be understood here in the sense of *submultiple;* it merely implies that the one varies as the other, and that both are expressed by the same number of degrees.

From what has now been said, it is plain that from having the number of degrees given, we could not actually construct the *arc*, because a degree of a circle is merely the 360th part of the circumference, without any limitation as to the length of that circumference, or as to whether the radius of the circle is an inch or a mile; but from having the number of degrees given, we *could* construct the *angle*, because an angle of one degree is the 360th part of a fixed and invariable magnitude, namely, four right angles. As it is desirable, in trigonometry, that the circular degree should be equally invariable with the angular degree, it is customary to assume a fixed length for the *trigonometrical radius*, and to represent it numerically by 1.

On the Trigonometrical Lines.

85. Angles themselves do not directly enter into trigonometrical computations, but certain straight lines connected with the angles, and with the arcs which subtend or measure them. These lines we shall now define.



SINE.—The sine of an arc, or of the angle which it subtends at the centre, is a line drawn from one extremity of the arc at right angles to the diameter from the other extremity. Thus Cm is the sine of the arc BC, or of the angle BAC. In like manner $C'm'$ is the sine of the arc BC', or of the angle BAC', which the arc measures.

If the angle be a right angle, as the angle BAD, the subtending arc BD is 90°:—it is called a *quadrant;* the sine of 90°, expressed in short thus, $\sin 90°$, is equal to the radius; and as radius is always regarded as numerically represented by unit, $\sin 90° = 1$. For any other arc or angle, the sine being less than radius, is numerically expressed by a proper fraction or decimal.

COSINE.—The cosine of an arc, or of the angle which it subtends, is the portion of a diameter intercepted between the centre and foot of the sine. Thus, Am is the cosine of the arc BC, or of the angle BAC. In like manner, Am' is the cosine of the arc BDC', or of the angle BAC'.

If the angle be a right angle, its cosine is 0; for Am becomes less and less, as C approaches D, and vanishes when C reaches D; that is, $\cos 90° = 0$.

On the other hand, Am increases as C approaches B, and becomes equal to radius when C reaches B, or when the arc BC becomes 0; hence $\cos 0° = 1$, but $\sin 0° = 0$.

Again, as C', the extremity of the arc BC', approaches B', the cosine Am' increases, and the sine, $C'm'$, diminishes; and when C' coincides with B', the arc becoming a semicircle, or 180°, the cosine becomes equal to radius, and the sine to 0; therefore $\sin 180° = 0$,

$\cos 180° = -1$. We put a *minus sign* before this 1 as a mark of *direction;* as $AB=1$, or $+1$, the notation $AB'=-1$, implies that AB, AB' are equal in *length*, but opposite in *direction*.

This is a very important appropriation of the opposite signs + and − to denote opposite directions. If Am were equal in length to Am', and we were to put p for the numerical expression for this length, we should say that $Am=p$, and that $Am'=-p$, or that $\cos BAC=p$, and $\cos BAC'=-p$; and, without referring to a figure, we should thus sufficiently indicate that the latter cosine is measured from A in a direction opposite to that of the former.

TANGENT, SECANT.—The tangent of an arc, or of the angle it subtends, is the straight line drawn from the beginning of the arc, B, and terminating in the prolongation of the radius or diameter, through the end of the arc. Thus, BT is the tangent of the arc BC, or of the angle BAC. In Geometry, *tangent* means simply a line that touches, without any reference to its length; but in Trigonometry, only so much of the touching line as is comprised between the beginning of the arc and the prolonged diameter through the end of it, is the tangent. And the line from the centre, thus limiting the tangent, is called the *secant* of the arc or angle. Thus, AT is the secant of the arc BC, or of the angle BAC. From these definitions, it is plain that of the arc BC', BT' is the tangent, and AT' the secant.

If BT, BT', be of the same length, and the numerical expression for that length be q, then $\tan BC=q$, and $\tan BC'=-q$; the opposite signs implying opposition of direction.

It is obvious that the smaller the arc, within the limits of 0° and 90°, the smaller will be the tangent and secant: as C approaches B, $\tan BC$ diminishes, and becomes 0 when BC becomes 0; but the secant, diminishing with the tangent, becomes AB or radius, when BC becomes 0.

On the other hand, when BC, by increasing, becomes equal to BD, or 90°, the tangent and secant both become infinite; we have therefore $\tan 0°=0,\ \tan 90°=\infty;\ \sec 0°=1,\ \sec 90°=\infty$
the mark ∞ being the sign for *infinite*.

As to the *sign* of the secant, you will observe that from the ever-varying direction which that line, radiating from the centre, takes, we cannot apply to *it* the distinctive signs + and −, hitherto employed to distinguish one direction from a direction opposite to it; secants never have directly opposite positions, with the single exception of the secants through D and D'; the former being +, and the latter −, agreeably to what has already been laid down as to the office of these directive signs. We shall say more about these signs and about their application to the secant hereafter.

COTANGENT, COSECANT.—A tangent drawn from D, the termination of the quadrant BD, and limited by the secant AT of the arc BC (or by its prolongation), is called the *cotangent* of the arc BC, or of the angle subtended by BC; thus, Dt is the cotangent of BC, or of the angle BAC, that is, $Dt=\cot\text{an} BAC$. In like manner, Dt' is the cotangent of the arc BC', or of the angle BAC'. And the part of the secant, or of its prolongation, intercepted between the centre and the cotangent is the cosecant; thus $At = \text{cosec} BAC$, and $At' = \text{cosec} BAC''$, AT'' being the secant of BAC', agreeably to the definition.

The six lines now defined, in connection with an arc or angle, deserve careful consideration, as it is with them, or rather with the numerical representatives of them, that all the computations respecting angles are performed. Before proceeding further, you should describe a circle, and having taken for C any point in the semicircumference BDB', you should draw, for the arc BC (and therefore for the angle BAC), all the six trigonometrical lines, whether C be to the right or to the left of D, taking care that each line conforms to the definition of it given above.

86. The cosine, cotangent, cosecant of an arc or angle, is so called, because, whatever is applied, either additively or subtractively, to an arc or angle, to reduce it to 90°, is regarded as the *complement* of that arc or angle; thus, to make up 90°, the arc CD must be applied additively to BC; CD is, therefore, the complement of BC; and the sine Cn of this complement—equal to mA—is the complemental sine, or, in short, the *cosine* of BC. Again, to reduce the arc BC'' to 90°, DC'' must be applied subtractively; DC'', therefore, thus taken negatively, is the complement of BC''; $C''n = m'A$, the sine of that complement, is the *cosine* of BC''. And generally, any co-line of an arc or angle is the line itself of the complement of that arc or angle.

As the term complement is applied to the difference between any arc or angle and 90°, so the term *supplement* is applied to the difference between the arc or angle and 180°; thus CB' is the supplement of BC, and $C''B'$ the supplement of BC''. A glance at the figure suffices to show that the sine of an arc is equal to the sine of its supplement; thus, if $BC = B'C''$, then $BC + BC'' = 180°$, and, therefore, BC'' is the supplement of BC; and $Cm = C''m'$. The cosine of an arc is also equal to that of its supplement, in *length*, but is opposite to it in *sign;* thus, Am, Am' are equal, but lie in opposite directions. In examining into the peculiarities of the trigonometrical lines connected with supplemental arcs, it is necessary to take the same point B for the origin or commencement of both the arc and its supplement, that is, to take BC'' for the supplement of BC, and not CB'.; the tangents of all the arcs are considered to originate at B, which implies that the arcs themselves are to originate there; by thus having one common extremity B for all the arcs, and one common extremity D for all the complements, all confusion and perplexity is avoided, and the signs of direction of the different trigonometrical lines become referable to a fixed and uniform system.

[It is, indeed, chiefly with a view to the establishment of such a uniform system of directive signs, that the arcs of a circle, in connection with the angles which they measure, have been introduced at all; although a reference to these measuring arcs would have been proper to justify and give significance to the terms *tangent* and *secant.* In several works on trigonometry, angles are considered independently of their measuring arcs, and sine and cosine, with their proper directive marks, being deduced by peculiar definition, the other trigonometrical lines are derived from *them;* but this mode of proceeding appears to be too abstract for a beginner. A diagram, addressed to the eye, seems to us to be an aid too valuable to be dispensed with ; and, moreover, the propriety of the names given to the trigonometrical lines cannot be seen without reference to the circle, in which, in fact, they all had their origin ; and besides all this, the terms sine, tangent, &c., as trigonometry, thus considered, defines them,—without, however, restricting the radius to the particular numerical value $r=1$—are terms too much needed in geometrical inquiries to be dispensed with.]

87. Referring again to the diagram at page 134, and merely keeping in remembrance the 47th proposition of Euclid's first book, and that the trigonometrical radius is numerically expressed by 1, we have, for any arc or angle A, the following properties :—

$$\sin^2 A+\cos^2 A=1 \quad \therefore \quad \sin A=\sqrt{(1-\cos^2 A)},\ \cos A=\sqrt{(1-\sin^2 A)}$$
$$1+\tan^2 A=\sec^2 A \quad \therefore \quad \sec A=\sqrt{(1+\tan^2 A)},\ \tan A=\sqrt{(\sec^2 A-1)}$$
$$1+\cot^2 A=\text{cosec}^2 A \quad \therefore \quad \text{cosec}\, A=\sqrt{(1+\cot^2 A)},\ \cot A=\sqrt{(\text{cosec}^2-1)}$$

And if, in addition to the above geometrical principle, we avail ourselves of the property that the sides about the equal angles of equiangular triangles are proportional (Euc. Prop. 4, B. vi.), we shall have, from comparing together the equiangular right-angled triangles CAm, TAB ; $C'Am'$, $t'AD$, the following equations, namely,

$$\left.\begin{array}{l} \dfrac{\sin A}{\cos A}=\dfrac{\tan A}{1} \quad \therefore \quad \tan A=\dfrac{\sin A}{\cos A} \\ \dfrac{\cos A^{*}}{\sin A}=\dfrac{\cot A}{1} \quad \therefore \quad \cot A=\dfrac{\cos A}{\sin A} \end{array}\right\} \quad \therefore \quad \cot A=\frac{1}{\tan A}$$

$$\frac{\sin A}{1}=\frac{1}{\text{cosec}\, A} \quad \therefore \quad \sin A=\frac{1}{\text{cosec}\, A}, \qquad \text{cosec}\, A=\frac{1}{\sin A}$$

$$\frac{\cos A}{1}=\frac{1}{\sec A} \quad \therefore \quad \cos A=\frac{1}{\sec A}, \qquad \sec A=\frac{1}{\cos A}$$

It thus appears that the following quantities are the *reciprocals* of one another, namely,

$\tan A$, $\cot A$; $\sin A$, $\text{cosec}\, A$; $\cos A$, $\sec A$.

And, moreover, that $\sin A$, $\cos A$ being known, all the other trigonometrical lines may be derived from *them*. From this circumstance, we may always arrive at the proper *sign* as well as at the proper numerical value of a secant and cosecant, from attending to the cosine and sine ; the *sign* of the secant is always the same as that of the cosine ; and the sign of the cosecant the same as that of the sine. The sine of an angle, whether acute or obtuse, is + ; but the cosine of an acute angle is +, and the cosine of an obtuse angle −.

88. The numerical values of the trigonometrical lines, for all values of the angle A, from $A=0°$ up to $A=90°$, are computed to several places of decimals, and arranged in *tables*. To compute for angles above 90° would be superfluous ; for the sine, tangent, &c., of an angle *above* 90°, is the same as the sine, tangent, &c., of an angle as much below 90° ; and the cosine, cotangent, &c., of an angle above 90°, is

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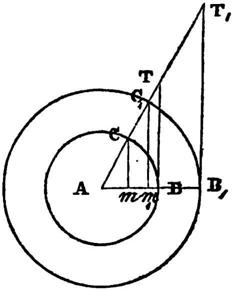
[*] It will, of course, not be overlooked that $C'm'=Cm$.

the same as the cosine, cotangent, &c., of an angle as much below 90°, as a mere inspection of the diagram is sufficient to show.

Observe that the values thus inserted in the tables are *numerical* values—strictly abstract *numbers;* they are not the lines themselves, but their numerical representatives, 1 being the numerical expression for the radius. On this account, what are called here the trigonometrical lines, are by some writers called merely *ratios;*[*] because, whatever length, r, be taken for the radius of the circle, in connection with which the lines are drawn, the *ratios* $\frac{\sin A}{r}$, $\frac{\cos A}{r}$, $\frac{\tan A}{r}$, &c. would remain the same; for

$$\frac{Cm}{AC}=\frac{C_,m_,}{AC_,};\ \frac{Am}{AC}=\frac{Am_,}{AC_,};\ \frac{BT}{AB}=\frac{B_,T_,}{AB_,};\ \text{\&c.}$$

so that if r be taken equal to the concrete *unit* of measure, the above ratios become the abstract numbers inserted in the tables under the heads of sin, cos, tan, &c. If, therefore, we ever wish to generalize any formula in which these quantities enter—that is, if we wish to remove the trigonometrical restriction, as to the particular value of the radius—we may always replace the abstract numbers $\sin A$, $\cos A$, &c., by the equal abstract numbers $\frac{\sin A}{r}$, $\frac{\cos A}{r}$, &c, r being any lineal value whatever.



TABLES: RIGHT-ANGLED TRIANGLES.

89. We shall here suppose that you have a book of Trigonometrical Tables before you; these usually consist of, 1st. A table of logarithms of numbers; 2nd. Of a table of the numerical values of the sines, cosines, &c., from 0° up to 90°; and 3rd. Of the logarithms of these numbers: the former is called a table of the *natural* sines, cosines, &c.; the latter a table of *logarithmic* sines, cosines, &c.[†]

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[*] How with any propriety they can be called "ratios of angles," as some call them, the author is at a loss to conceive; they are evidently ratios of *lines*, not of *angles*.

[†] There are several collections of Trigonometrical Tables suitable for use in connection with this work; among the cheapest and most compact of these, are the Tables published by the Society for the Diffusion of Useful Knowledge, revised by Professor De Morgan; the Tables in Chambers's Educational Course, and the volume of Tables by the author of this Treatise. In the latter two works, the decimals are carried to seven places, an extent sufficient for even the most delicate operations of trigonometry, and greater than necessary for ordinary purposes; in some of the calculations to be hereafter given, only five or six places of decimals will be employed. a larger number being superfluous.

But as the natural sines and cosines all range between 0 and 1, every sine and cosine between these extreme limits must be less than unity; and as the logarithms of numbers less than 1 are all negative (Algebra, p. 100), it is found convenient, for the purpose of avoiding negative logarithms, to increase the logarithm of every sine, cosine, &c., by 10. This amounts, in fact, only to a change in the numerical value of the trigonometrical radius, making it 10^{10} instead of 1; so that, while the natural sines, cosines, &c., are computed to $r=1$, the logarithmic sines, cosines, &c., are computed to $r=10^{10}$; the ratio of the *lines*, $\frac{\sin A}{r}$, $\frac{\cos A}{r}$, remaining, however, the same for either value, as shown in the preceding page.

The tables to which we here refer clearly furnish the following valuable particulars:—

The table of natural sines, &c., supplies, 1st. The numerical values of the perpendicular Cm and base Am of every possible right-angled triangle ACm, of which the hypotenuse AC is 1; these are entered in the table under the heads of sine and cosine.

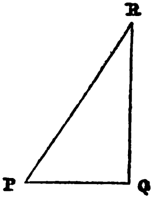
It supplies, 2nd. The perpendicular BT and hypotenuse AT of every right-angled triangle ATB, of which the base AB is 1.

It is impossible, therefore, that any right-angled triangle can be met with in practice the exact likeness of which is not to be found, in numerical terms, in the table; of every such triangle, there are, as we have just observed, *two* exactly similar to it, and of each of which all the sides are ready computed.

The table of logarithmic sines, &c., furnishes the logarithms of the numerical values of the sides of each of the *two* triangles just mentioned, when those sides are taken 10^{10} times as great: in the former table, we have the sides of each of two triangles, similar to any proposed triangle computed to the scale AB or $AC=1$; in the latter table, we have the logarithms of those sides computed to the scale AB or $AC=10^{10}$.

90. In order to convey a notion of the practical use of these tables, we shall take an example.

1. A distance PQ of 200 feet is measured in a direct line from the bottom of a steeple, the angle of elevation P, of the top R, is 47° 30′; it is required to find the height QR of the steeple.



By the diagram at page 134, we know that in the tables there are two triangles AmC, ABT, similar to the triangle PQR already computed for us; from the first of these we have,

$$Am : mC :: PQ : QR = PQ\frac{mC}{Am}; \text{ that is, } QR = PQ\frac{\sin P}{\cos P}$$

$$= PQ \tan P, \text{ so that } QR = 200 \tan 47° \ 30',$$

All the tables referred to are accompanied by introductory explanations of the mode of using them, so that any instructions respecting the use of the tables would be unnecessary in the present work.

or comparing our triangle PQR with the second of the tabular triangles ABT, we have, at once, since

$$PQ=200 \text{ times } AB,\ QR=200\,BT=200\tan 47^\circ\ 30'.$$

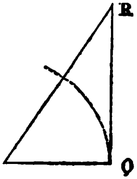
Either of these operations being performed, we have then only to look for tan 47° 30′ in the table of natural tangents, and to multiply the number there supplied by 200; we thus get $QR=200\times\cdot 10913=218\cdot 26$ feet.

It is plain that the second of the two tabular triangles offered to our choice is the preferable, and *in all cases*, it is better to take that one of the two triangles in which the unit-line or *radius* corresponds to the *given line* in the triangle proposed; for representing this given line by a it is a times the unit-line, and, therefore, either of the other sides of the proposed triangle is a times the corresponding tabular side.

2. Suppose, instead of the height QR being required, the length PR of a rope from R to P be wanted.

Taking the tabular triangle in which the unit-side, or radius, corresponds to the given side PQ, we see that it is the triangle ABT; and that since $PQ=200$ times AB, therefore $PR=200\,AT$, that is,

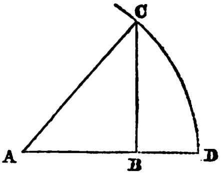
$$PR=200\sec 47^\circ\ 30'=200\times 1\cdot 4802=296 \text{ ft.}$$



Without referring every time to the diagram at page 134, it is better to take the given side as a radius, and the vertex of the given angle as a centre, and to describe an arc, as in the above figure, or rather to *imagine* such arc described, because we shall then see at once whether the line in the table, corresponding to that we want, takes the name of tangent, or secant, or sine, &c. For example—

3. Suppose the hypotenuse AC is 288, the angle A 50° 54′, and that it is required to find both BC and AB.

As AC is the given side, we imagine the arc CD described from centre A, with radius AC; and we then at once see that CB corresponds to *sine*, and AB to cosine, of the angle A; hence,



$$BC=AC\sin A,\quad AB=AC\cos A$$

that is, $BC=288\sin 50^\circ\ 54'=288\times\cdot 776046=223\cdot 5$

$AB=288\cos 50^\circ\ 54'=288\times\cdot 630676=181\cdot 63$

91. The tables referred to in these solutions are the tables of *natural* sines, cosines, &c., and for the solution of right-angled triangles it will, in general, be found that by the use of these tables the work is shortened; you have seen from the foregoing examples that in each case *one* reference to the tables was sufficient, whereas, if the logarithmic tables had been employed, *three* references would have been necessary.

You must never forget that in using the logarithmic tables of sines, cosines, &c., each tabular logarithm is increased by 10. If the number of these tabular logarithms to be *added* be the same as the number to be *subtracted*, the superabundant 10's, of course, destroy one another, and no correction is necessary; but, in the contrary case, as many 10's must be subtracted from, or added to, the result, as the additive or subtractive sets exceed, one set the other, in number. Let us return to the first of the preceding examples; the logarithmic operation is as follows:—

The formula of solution is $QR=200 \tan 47° 30'$; so that, in logarithms, we have $\log QR=\log 200+\log \tan 47° 30'$; and as there is here one trigonometrical quantity to be added, and none to be subtracted, the correction 10 must be applied, subtractively, to the result. It is usual to arrange the work thus:

	Logarithms.
	−10
200....	2·301030
tan 47° 30'.....	10·037948
$QR=218·26$....	2·338978

Again, take Ex. 3 above; that is, the expressions

$$BC=288 \sin 50° 54', \text{ and } AB=288 \cos 50° 54'$$

	Logarithms.		Logarithms.
	−10		−10
288....	2·459393	288....	2·459393
sin 50° 54'.....	9·889888	cos 50° 54'....	9·799806
$BC=223·5$....	2·349281	$AB=181·63$....	2·259199

Hitherto an angle and a side have been given to find a side; suppose now that two sides are given to find an angle; for example, in the triangle PQR, page 140, let PQ, PR be given to find the angle P. With the vertex of this angle as centre, and with a radius equal to *either* of the given sides, as PQ for instance, imagine an arc to be described, and notice what trigonometrical name the other given side takes, in reference to this radius; you see that it is *secant*. You know therefore that $PR=PQ \sec P$; and, consequently, that $\sec P=\frac{PR}{PQ}$. Or if, instead of the given base, you take the given hypotenuse for radius—as AC, in the triangle ABC, page 140,—then, in reference to this radius, the base AB will evidently be *cosine;* therefore $AB=AC \cos A$; $\therefore \cos A=\frac{AB}{AC}$. In the first way of proceeding, we found that the secant of the angle at the base was hypotenuse divided by base; in the second way, that the cosine of the angle at the base was base divided by hypotenuse; the one result being the reciprocal of the other, as it ought to be, since, as shown at page 137, $\cos A=\frac{1}{\sec A}$.

And generally, when two sides are given to find an angle, take the vertex of that angle as centre, make one of the given lines, terminating in that vertex, radius, and imagine an arc described. Observe what trigonometrical name the other given side takes, *that* will be the trigonometrical name under which the sought angle is to be found in the tables; the tabular number being *always* equal to that given side, *not* taken for radius, divided by that which *is* taken for radius. This is a general rule, the truth of which you may fairly be expected to see for yourself, from the explanations already given. But it may be well to give to the rules of operation a more formal enunciation.

Rules for the Solution of Right-angled Triangles.

92. When one of the acute angles of a right-angled triangle is given, the other is also given; and when one is determined, the other is also determined, for the two angles are the complements of one another; if one be A, the other is $90° - A$. This must be borne in mind in reading the following rules.

1. *When an angle and a side are given.*

Rule 1. With the vertex of the given angle as centre, and the given side terminating in that vertex, as radius, imagine an arc described, and observe the *trigonometrical name* that the sought side takes.

2. Enter the tables with the given angle, and take out the tabular number under this name; multiply that number by the length of the given side, and the product will be the length of the sought side.

2. *When two sides are given to find an angle.*

Rule 1. With the vertex of the sought angle as centre, and a given side terminating in that vertex, as radius, imagine an arc described, and observe what *trigonometrical name* the other given side takes.

2. Divide the latter side by the former; the quotient will be the tabular number of *the same* name, under which the value of the sought angle will be found.

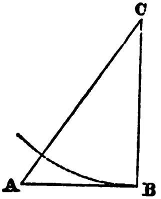
Note 1. If two sides of a right-angled triangle are given to find the *third* side, then, without first finding the acute angles, we may at once apply Prop. 47 of Euclid, B. i.; thus, if a, b, be the perpendicular and base, and c the hypotenuse, then

$$c = \sqrt{(a^2 + b^2)},\quad a = \sqrt{(c^2 - b^2)} = \sqrt{\{(c+b)(c-b)\}}$$
$$b = \sqrt{(c^2 - a^2)} = \sqrt{\{(c+a)(c-a)\}}$$

2. When logarithmic sines, cosines, &c., are employed in the application of the preceding rules, it must always be recollected that every tabular number is increased by 10 (see p. 139), so that 10 must be subtracted for every such tabular number added and added for every one subtracted. But, in working by the above rules, a logarithmic sine, cosine, &c., will never have to be *subtracted*; unless, in the case in which a secant enters, you may choose to replace $\sec A$ by $\frac{1}{\cos A}$, which is sometimes done; that is, to substitute for $\log \sec A$ its equivalent $-\log \cos A$; by which an additive quantity is replaced by a subtractive one; the former, corrected, is $\log \sec A - 10$, the latter, $-\log \cos A + 10$; the two results being the same. On account of this ready way of converting the secants into cosines, in the more portable books of tables, the column of secants is omitted.

93. We shall now give an example or two of the application of the preceding rules.

(1) In the right-angled triangle ABC are given the angle A at the base and the perpendicular BC to find the base AB; namely, $A=64^\circ\ 30'$, $BC=120$ feet.



As A is given, the vertical angle C is given, namely, $C=90^\circ-64^\circ\ 30'=25^\circ\ 30'$; therefore, taking C as centre and CB as radius, we see that AB becomes *tangent*.

$$\therefore\ AB=BC\tan C=120\tan 25^\circ\ 30'=120\cot 64^\circ\ 30'$$

Without logarithms.

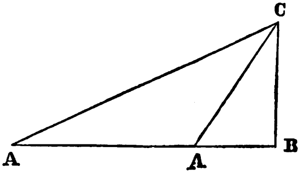
$$\begin{array}{lr} \tan 25^\circ\ 30'= & \cdot 47698 \\ & 120 \\ \cline{2-2} AB= & 57\cdot 2376 \text{ feet.} \\ \cline{2-2} \end{array}$$

By logarithms.

$$\begin{array}{lr} & -10 \\ 120\ldots\ldots & 2\cdot 07918 \\ \tan 25^\circ\ 30\ldots\ldots & 9\cdot 67850 \\ \cline{2-2} AB=57\cdot 238\ldots\ldots & 1\cdot 75768 \\ \cline{2-2} \end{array}$$

It is plain that it was unnecessary to take the complement of A; for the tangent of C being the cotangent of A, the form to compute from may be written at once $AB=BC\cot A$; and similarly in other cases.

(2) The distance between two objects at A, A', from the nature of the intervening ground, cannot be measured; but A, A' are in the same straight line with B, the bottom of a tower 120 feet high, and the angles of elevation of the top C, as taken at A and A', are found to be 33° and 64° 30'; required the distance AA'.



1. In the triangle ABC are given $BC=120$, $A=33^\circ$.

$$\therefore\ AB=BC\cot A=120\cot 33^\circ.$$

2. In the triangle $A'BC$ are given $BC=120$, $A'=64^\circ\ 30'$.

$$\therefore\ A'B=BC\cot A'=120\cot 64^\circ\ 30'$$

$$\therefore\ AA'=AB-A'B=120(\cot 33^\circ-\cot 64^\circ\ 30').$$

Without logarithms.

$$\begin{array}{lr} \cot 33^\circ= & 1\cdot 53987 \\ \cot 64^\circ\ 30'= & \cdot 47698 \\ \cline{2-2} & 1\cdot 06289 \\ & 120 \\ \cline{2-2} AA'= & 127\cdot 5468 \text{ feet.} \\ \cline{2-2} \end{array}$$

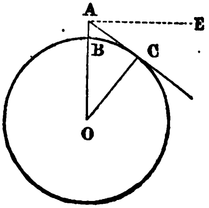
By logarithms.

	−10		−10
120....	2·07918	120....	2·07918
cot33°....	10·18748	cot64° 30′....	9·67850
AB=184·79....	2·26666	$A'B$=57·24....	1·75768

$$\therefore AA'=184{\cdot}79-57{\cdot}24=127{\cdot}55$$

Hence the required distance is 127·55 feet.

(3) From the top of a mountain three miles high, the angle of depression EAC of the remotest visible point of the surface of the sea was taken, and found to be 2° 13′ 27″; it is required from this to determine the diameter of the earth.



Let O be the centre of the earth, AB the mountain, AC the line drawn from A to touch the surface of the earth in C, and consequently the angle EAC, the angle of depression, below the horizontal line AE.

Then EAC is the complement of OAC; but the angle O is also the complement of OAC, since C is a right angle (Euc. Prop. 18, B. iii.), therefore, $O=EAC=2^\circ\ 13'\ 27''$.

BA is the difference between the secant of the angle O and the radius OB; the difference between the *trigonometrical* secant of the angle O, and the radius 1, is given by the table of natural secants, and this tabular difference, multiplied by the radius OB, must be equal to the difference AB; since a line, computed to any radius, is equal to the corresponding trigonometrical line multiplied by that radius.

$$\therefore AB=(\sec O-1)\,OB \quad \therefore OB=\frac{AB}{\sec O-1}$$

hence we have only to divide three miles by the natural secant of 2° 13′ 27″ diminished by 1: the quotient will be found to be 3979 miles.

We may dispense with a table of secants by substituting $\frac{1}{\cos O}$ for $\sec O$; we shall thus have $OB=\frac{AB\cos O}{1-\cos O}$

Neither of these expressions for OB is fit for logarithmic computation; for, by *logarithms*, the simplest operations that can be performed are multiplication and division. It will, however, be shown hereafter that $1-\cos O=2\sin^2\frac{1}{2}O$, so that the last expression may be put under the form $\frac{AB\cos O}{2\sin^2\frac{1}{2}O}$, which, as it involves no addition or subtraction operation, may be worked by logarithms: the logarithmic form for $2OB$ will be

$$\log 2OB = \log AB + \log \cos O - 2 \log \sin \frac{1}{2} O + 10$$

the 10 being introduced, as an additive correction, because there is only one additive cosine, but two subtractive sines, namely,

$$\sin \frac{1}{2} O + \sin \frac{1}{2} O$$

Examples for Exercise.

(1) In a right-angled triangle are given the base, 288 feet, and the angle at the base 53° 8′; required the perpendicular and the hypotenuse.

(2) From the edge of a ditch, surrounding the walls of a fort, the angle of elevation of the top of the wall was taken, and found to be 62° 40′; required the length of a scaling-ladder to reach from the edge of the ditch—which is 36 feet wide—to the top of the wall.

(3) A person on one side of a river takes the angle of elevation of an obelisk standing on the other side, and finds it to be 55° 54′; then going back 100 feet, in the same straight line with the foot of the obelisk, he finds the angle of elevation there to be 33° 20′; what was the height of the obelisk?

(4) The mountain called the Peak of Teneriffe is nearly 2½ miles high; the angle formed at the top between a plumb-line, which is of course vertical, and the line conceived to touch the earth in the remotest visible point is 87° 58′; suppose the height of the peak to be exactly 2½ miles, what will be the diameter of the earth?

(5) If the top of a mountain 6,600 feet high can just be seen at sea 100 miles off, what must be the diameter of the earth?

(6) The angles of elevation of a balloon were taken at the same time by two observers, both on the same side of it and in the same vertical plane as the balloon; the angles were found to be 35° and 64°, and the observers were 880 yards apart; required the height of the balloon.

(7) A ladder 40 feet long can be so placed that it shall reach a window 33 feet from the ground, on one side of the street; and upon being turned over, the foot of it remaining in the same place, it will reach a window on the opposite side 21 feet from the ground; required the breadth of the street.

(8) The base of a right-angled triangle is 162 feet, and the angle at the base 53° 7′ 48″; required the perpendicular and hypotenuse.

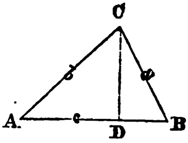
(9) The perpendicular and hypotenuse of a right-angled triangle are respectively 216 and 270 feet; required the angle at the base.

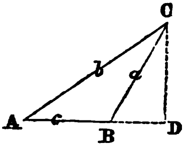
(10) At the top of a castle standing on a hill by the sea-shore, the angle of depression of a ship at anchor was observed to be 4° 52′; at the bottom of the castle, the angle of depression was 4° 2′; required the horizontal distance of the ship, as also the height of the hill on which the castle stands, the height of the castle itself being 60 feet.

Oblique-angled Triangles.

94. Every triangle is said to have *six parts:*—the three sides and the three angles. If any three of these be given, provided they be not the three angles, the remaining three may be determined by calculation, as we are now to show.

Let ABC be any triangle; let the angles be represented by the capital letters A, B, C, and the sides opposite to these by the corresponding small letters a, b, c.





From either vertex, as C, let a perpendicular CD be drawn to the opposite side c; this perpendicular will fall within the triangle, if the angle B be acute, and without it, if this angle be obtuse; in either case ADC, BDC will be right-angled triangles; from each of these right-angled triangles we may deduce an expression for the common perpendicular CD, in terms of the hypotenuse and angle at the base, by the preceding rule; thus,

From triangle ADC, $CD = b \sin A$

„ „ BDC, $CD = a \sin B$

for the sine of an angle is the same as the sine of its supplement;

$$\therefore\ a \sin B = b \sin A \quad \therefore\ \frac{a}{b} = \frac{\sin A}{\sin B}$$

Converting this equation into a proportion (Algebra, p. 68), we have

$$a : b :: \sin A : \sin B$$

which furnishes an easy rule for the solution of what is called the first case of oblique-angled triangles.

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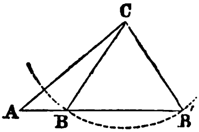
95. Case 1.—*When of the three given parts two are opposite to each other, namely an angle and a side.*

Rule. As one side of a triangle is to another, so is the sine of the angle opposite to the former, to the sine of the angle opposite to the latter.

Note.—This rule is of course as applicable to right as to oblique-angled triangles. In the preceding articles the tables of natural sines, cosines, &c. were employed, as well as the logarithmic; and the work by the former tables was, in general, found to be the shorter. But in working a case of oblique-angled triangles, the logarithmic tables will be almost exclusively used, as the operation, in reference to such triangles, is usually more easy when wholly conducted by logarithms. Under particular circumstances, however, the *third* case of oblique-angled triangles may be conveniently solved without logarithms, as will be hereafter shown.

This first case of oblique-angled triangles is what is called the *ambiguous case*, when the given parts are two sides and an angle opposite to one of them. For the angle opposite to the other, being indicated in the rule by its *sine*, we may be at a loss to know whether the angle itself is acute or obtuse; for the same *sine* equally belongs to an angle and its supplement. Other particulars concerning the triangle in hand are sufficient to overrule our choice, and if these are furnished, the ambiguity is removed; but, in the absence of all information as to the shape of the triangle under examination, we have nothing to guide us in the selection, beyond the two geometrical truths that *two* angles of a triangle cannot be obtuse, and that the greater side must have the greater angle opposite to it: but Ex. 2, following, will sufficiently illustrate the ambiguity here adverted to. You will remember that Euclid says nothing about two triangles being equal, when the three things, the same in both, are two sides and an *opposite* angle; and his omission arose from the geometrical ambiguity corresponding to that implied in trigonometry. The annexed diagram shows that the two *unequal* triangles, ABC, $AB'C$ have two sides, AC, BC and the angle A, in one, the same as the two sides AC, $B'C$ and the angle A in the other, provided $BC = B'C$.



(1) In the triangle ABC are given

$A = 23° 45'$, $C = 17° 15'$, and $c = 60$ feet, to find a.

Here two angles, and a side opposite to one, are given to find the side opposite to the other; hence by the rule,

As $\sin C$, $17° 15'$ .... 9·472086 *to be subtracted.*

: $\sin A$, $23° 45'$ .... 9·605032

:: $c = 60$ .... 1·778151

: $a = 81·488$ ...... 1·911097

96. There is evidently an inconvenience in thus having to add together two logarithms, and to subtract a third. There is an ingenious way of avoiding this inconvenience, and of rendering the logarithms all additive; we shall explain it here and urge it upon your attention.

Suppose we had omitted the subtractive logarithm altogether; we should then have committed an error of excess in the final result, to the amount of 9·472086. This number is short of 10 by ·527914, so that if we not only cause the result to err in excess by 9·472086, but further increase the error by *adding* ·527914 to the other two logarithms, the error in the result will then be exactly 10; an error which is very easily allowed for, and precluded from appearing by actually suppressing 10 in the total amount. It is this that is always done in practice; instead of writing down a subtractive logarithm in the same column with additive logarithms, we write down what the subtractive log wants of 10. This is called the *arithmetical complement* of the log; and by taking this arithmetical complement *additive*, the sum of the whole column becomes too great by 10, which 10 is therefore suppressed in the result.

The arithmetical complement may be written down, when the number is before us, nearly as rapidly as the number itself. The best way is to commence with the first figure on the *left*, to subtract it, and every following figure from 9, till we come to the last significant figure, which we must subtract from 10. This is evidently the same, as to result, as commencing the subtraction in the usual way, that is, at the last figure, and proceeding, from right to left, to the first figure: the carrying 1 every time, and the uniform subtraction from 10, is the same as to omit the carryings, and subtract all, after the last figure, from 9.

When, therefore, having turned to the tables, we find log sin 17° 15' $=9 \cdot 472086$, we point to the 9, and write 0; to the 4, and write 5; to the 7, and write 2; to the 2, and write 7; to the 0, and write 9; to the 8, and write 1; and finally to the 6, and write 4. Instead of the above, therefore, the operation would be this,

As sin C, 17° 15'.......... ·527914 (*Arith. Comp.*)
: sin A, 23° 45'........ 9·605032
:: c, 60 .... 1·778151

: $a=81 \cdot 488$ ...... 1·911097 (10 *suppressed*).

And this should be the mode of proceeding in all cases in which a subtractive logarithm is to be combined with others that are additive.

(2) In the triangle ABC are given,

$a=345$, $b=232$, $B=37°\ 20'$ to find all the other parts.

1. To find A.

As b, 232 .... 7·634512 (*Arith. Comp.*)
: a, 345 .... 2·537819
:: sin B, 37° 20'.......... 9·782796

: sin A, 64° 24'.......... 9·955127 (*Suppress* 10).

As the sine of an angle is the same as the sine of its supplement, we have no more right to pronounce the angle A to be 64° 24', than to affirm it to be 180° − 64° 24' = 115° 36'. The case, therefore, is *ambiguous*. There are *two* triangles to which the given conditions apply: in one the angle A is 64° 24', and in the other it is 115° 36'. In actual practice a glance at the shape of the triangle will in general be sufficient to indicate whether the sought angle is acute or obtuse, and thus to remove the ambiguity; but if, as above, the mere measurements alone are given, and no information afforded as to whether the triangle is acute-angled or obtuse-angled, we have no grounds for preferring one to the other, and must therefore, compute for *both*. If the given angle B had itself been obtuse, or if the side opposite to the given *acute* angle B had been greater than that opposite to A, then, of course, there could have been no hesitation about the character of the other angles:—each of them must have been *acute*. As two angles A, B, of the proposed triangle, or rather, of the pair of triangles are now known, we can get the third angle, C, by subtracting the sum of those two from 180°; thus :—

2. To find C.

$$\begin{array}{rrcr}
A= & 64^\circ\ 24' & \text{or} & 115^\circ\ 36' \\
B= & 37^\circ\ 20' & & 37^\circ\ 20' \\
\hline
A+B= & 101^\circ\ 44' & \text{or} & 152^\circ\ 56' \\
A+B+C= & 180^\circ & & 180^\circ \\
\hline
\therefore\ C= & 78^\circ\ 16' & \text{or} & 27^\circ\ 4' \\
\hline
\end{array}$$

3. To find c.

$$\begin{array}{rlrcrl}
\text{As} & \sin A\ldots\ldots & \cdot044873 & & \cdot044873 & (\textit{Arith. Comp.}) \\
: & \sin C\ldots\ldots & 9\cdot990829 & \text{or} & 9\cdot658037 & \\
:: & a\ldots\ldots & 2\cdot537819 & & 2\cdot537819 & \\
\hline
: & c\ldots\ldots & 2\cdot573521 & \text{or} & 2\cdot240729 & \\
\hline
\end{array}$$

$$\therefore\ c=374\cdot56, \text{ or } 174\cdot07$$

consequently, the six parts of the triangle are either

$$a=345,\ b=232,\ c=374\cdot56;\ A=64^\circ\ 24',\ B=37^\circ\ 20',\ C=78^\circ\ 16'$$

$$\text{or } a=345,\ b=232,\ c=174\cdot07;\ A=115^\circ\ 36',\ B=37^\circ\ 20',\ C=27^\circ\ 4'.$$

Examples for Exercise.

NOTE.—When an angle is to be found, it will always be unambiguous when either of the other angles is obtuse, as also when the side opposite an acute angle is *greater* than the side opposite the angle to be found; the sought angle will then always be acute (Euclid, Prop. 18, B. i.). When neither of these conditions are stated, the sought angle may have either of two values, the one being the supplement of the other.

(1) Given $A=74^\circ\ 14'$, $B=49^\circ\ 23'$, and $c=408$, to find a and b.

(2) Given $b=153$, $c=137$, and $B=78^\circ\ 13'$, to find the remaining parts A, C, a.

(3) A person on one side of a river, wishing to know the distance of an object on the other side, measured a straight line of 200 yards by the river side, and having fixed up a staff at each end, measured with a theodolite the horizontal angles at each end of the line, between the other end and the object. These angles were 68° 2′, and 73° 15′; required the distances of the object from each end of the line.

(4) Two observers, 1,000 yards apart, take at the same time the angles of elevation of a balloon, somewhere between them in the same vertical plane. The angles were 36° and 54°; required the height of the balloon.

(5) Two ships of war intending to cannonade a fort, are kept so far from it by the shallowness of the water that they suspect the guns cannot reach it. In order, therefore, to discover its distance, the ships separate from each other a quarter of a mile, or 440 yards; from each ship

is then measured the angle which a line joining the other ship with the fort subtends. These angles are 83° 45′ and 85° 15′; required the distance between each ship and the fort.

(6) Given two sides of a triangle equal to 216 and 117 respectively, and the angle opposite to the latter, 22° 37′; required the angle opposite to the former to the nearest second.

---

97. Case 2.—*When the three given parts are two sides and the included angle.*

Let ABC be any plane triangle; the sides AC, BC, and the included angle C, are supposed to be given to find the angles A and B.

From the longer of the two given sides, CA, cut off a part CD equal to the shorter CB; draw BD, and perpendicular to it draw CE which produce to F, and draw EG parallel to AB.

Then because the right-angled triangles CEB, CED have the angles at B and D equal (Euclid, 5 of i.), and the side CE common, CE bisects the angle at C, and also the line BD. And because $DE=EB$, $\therefore DG=GA$ (Euclid, 2 of vi.); therefore, since CD is the half of $BC+CD$, and DG the half of DA, $\therefore CG=\frac{1}{2}(BC+CA)$; that is,

$CG=$half of the sum of the given sides;

and since AD is the difference of those sides, therefore,

$AG=$half the difference of the given sides.

Again, since the three angles of every triangle make two right angles, the sum of the angles CDB, CBD must be equal to the sum of the angles A, B of the proposed triangle, so that CBD is half that sum; therefore,

$CBD=$half the sum of the unknown angles;

also $ABD=$half their difference;

because the half sum added to the half difference must make up the greater of the two (Algebra, p. 12).

These preliminaries being settled, we have, by right-angled triangles,

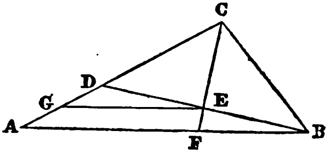
$CE=BE\tan CBD$,
$EF=BE\tan ABD$

$\therefore CE : EF :: \tan CBD : \tan ABD$

But (Euc. 2 of vi.)

$CE : EF :: CG : GA$

$\therefore 2CG : 2GA :: \tan CBD : \tan ABD$



and this proportion expressed in words is the following rule.

Rule. As the sum of the two given sides is to their difference, so is the tangent of half the sum of the opposite angles to the tangent of half their difference.

that is, $$\frac{a+b}{a\sim b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A\sim B)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A\sim B)}$$

(1) Two sides of a triangle are 345 and 174·07; and the angle between them 37° 20′; required the other parts.

$$\begin{array}{rlcrl}
a = & 345 & \qquad & A+B+C = & 180^\circ \\
b = & 174{\cdot}07 & & C = & 37^\circ\ 20' \\
\hline
a+b = & 519{\cdot}07 & & A+B = & 142^\circ\ 40' \\
a-b = & 170{\cdot}93 & & &
\end{array}$$

It remains therefore now to find $A-B$, by the above rule.

As $a+b$, 519·07.... 7·284774 (*Arith. Comp.*)

: $a-b$, 170·93...... 2·232818

:: $\tan\frac{1}{2}(A+B)$, 71° 20′....10·471298

: $\tan\frac{1}{2}(A-B)$, 44° 16′.... 9·988890

Adding, $A = 115^\circ\ 36'$, the greater angle.

Subtracting, $B = 27^\circ\ 4'$, the less angle.

The angles opposite to the two given sides being now found, we may find the third side by the former rule; thus, taking instead of the angle A. its supplement 64° 24′, as the sines are the same:—we prefer to take this rather than the angle B, because the log 345 is more readily got than log 174·07—we have

As $\sin A$, 64° 24′.......... ·044874 (*Arith. Comp.*)

: $\sin C$, 37° 20′........ 9·782796

:: a, 345 .... 2·537819

: c, 232 .... 2·365489

Examples for Exercise.

(1) Given two sides of a triangle, 23·859 and 29·271, and the angle between them, 40° 56′, to find the third side.

(2) Given two sides of a triangle, 120 yards and 112 yards, and the included angle 57° 57′; required the other parts.

(3) In a triangle ABC are given.

$$AB = 345,\ AC = 232,\ A = 27^\circ\ 4';$$

required the angles B and C.

(4) An obelisk stands on the top of a smooth declivity: if a distance of 40 feet be measured down this declivity from the bottom of the obelisk, the angle of elevation of the top will be 41°; but if the measuring be continued 60 feet further, the angle of elevation will be only 23° 45′; required the height of the obelisk.

(5) Given $a = 16{\cdot}9584$, $b = 11{\cdot}9613$, and $C = 60^\circ\ 43'\ 36''$, to find the side c.

---

98. CASE III. *When the given parts are the three sides.*

Referring to the two triangles at page 146, we have, by Euclid Propositions 12 and 13 of Book ii., When B is acute, $AC^2=BC^2+BA^2-2BA\cdot BD$

When B is obtuse, $AC^2=BC^2+BA^2+2BA\cdot BD$

But $BD=BC\cos CBD$; therefore by substitution, putting a, b, c, for the three sides of the triangle, we have

When B is acute, $b^2=a^2+c^2-2ac\cos B$

When B is obtuse, $b^2=a^2+c^2+2ac\cos CBD$

But CBD being the supplement of B, in the latter case, its cosine is $-\cos B$; hence, whatever be the character of the angle B,

$$b^2=a^2+c^2-2ac\cos B$$

Here one side of a triangle is expressed in terms of the other two and the included angle, and when the two sides are small numbers this formula may be conveniently used instead of the rule in next page, the natural cosine of the included angle being employed. As b is *any* side of the triangle, we of course have similar expressions for a^2 and b^2. Hence, for the cosines of the three angles in terms of the sides, we have

$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$

$$\cos B=\frac{a^2+c^2-b^2}{2ac}$$

$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

These formulæ are easily remembered, and they serve very well for the purpose when the sides are expressed by small numbers, but they are not adapted to logarithmic computation, on account of the addition and subtraction operations which they involve. A formula fitted for the application of logarithms must be free from additions and subtractions; the *denominator* of each expression is in a suitable form:—Let us try to convert the numerator into *factors*.

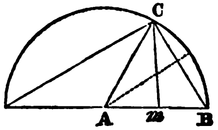
Subtracting each member of the first equation from 1, we have

$$1-\cos A=\frac{a^2-(b^2-2bc+c^2)}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

As the numerator is now the difference of two squares, we can convert it into factors (Algebra, p. 27), so that

$$1-\cos A=\frac{(a-b+c)(a+b-c)}{2bc}\ldots\ldots(1)$$

The expression $1-\cos A$, that is, the difference between radius and the cosine of an angle, is called the *versed-sine* of that angle; thus, mB is the versed-sine of the angle CAB, to the radius AB, regarded here as *unit.* But versed-sines are not computed in the ordinary tables; we must try therefore to change $1-\cos A$ into some equivalent trigonometrical value to be found in the tables.



Now the angle DCB, being in a semicircle, is a right-angle; and therefore (Euclid, Prop. 8, Book vi.) $BD \cdot Bm = BC^2$; but BC, the *chord* of the arc BC, is evidently twice the sine of half that arc, or twice the sine of half the angle A, which it measures, and BD, being twice the trigonometrical radius, is 2;

$$\therefore\ 2(1-\cos A) = (2\sin\tfrac{1}{2}A)^2$$

$$\therefore\ (1),\qquad \sin^2\tfrac{1}{2}A = \frac{\frac{1}{2}(a-b+c)\cdot\frac{1}{2}(a+b-c)}{bc}\ \ldots\ldots(2)$$

For brevity, put $s=\frac{1}{2}(a+b+c)$, half the sum of the sides

$$\therefore\ s-b=\tfrac{1}{2}(a-b+c)$$

$$s-c=\tfrac{1}{2}(a+b-c)$$

$$\therefore\ (2),\qquad \sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}\ \ldots\ldots\ldots\ldots(3)$$

a formula well adapted to logarithmic computation.

By means of this formula—and the two similar ones for $\sin\frac{1}{2}B$, $\sin\frac{1}{2}C$, which may be written down from a mere inspection of that for $\sin\frac{1}{2}A$, since the small letters always refer to the including sides—we may readily find the sines of the half angles from having the sides of the triangle given; it is useful to have also formulæ for the cosines of the half angles. Now since

$$\begin{aligned}\tfrac{1}{2}(1-\cos A) &= \sin^2\tfrac{1}{2}A\\ 1 &= 1\end{aligned}$$

By subtraction, $\frac{1}{2}(1+\cos A) = 1-\sin^2\frac{1}{2}A = \cos^2\frac{1}{2}A$

Hence, instead of subtracting each member of the equation above from 1, if we add 1 to each, we shall have

$$1+\cos A = \frac{(b^2+2bc+c^2)-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc} = \frac{(a+b+c)(b+c-a)}{2bc} = 2\cos^2\tfrac{1}{2}A$$

$$\therefore\ \cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}\ \ldots\ldots(4)$$

And dividing (3) by (4),

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\ \ldots(5)$$

Either of the formulæ (3), (4), (5), will enable us to find the half angles when the sides are given; and a formula for the whole angles may be easily obtained thus:

$$\begin{aligned}\text{since}\quad 1-\cos A &= 2\sin^2\tfrac{1}{2}A\\ \text{and}\quad 1+\cos A &= 2\cos^2\tfrac{1}{2}A\end{aligned}$$

by multiplying, $\sin^2 A = 4\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A$

$$\therefore\ \sin A = 2\sin\tfrac{1}{2}A\ \cos\tfrac{1}{2}A$$

$$\therefore\ (3)\text{ and }(4),\qquad \sin A = \frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}}\ \ldots(6)$$

But, for the object proposed, the formulæ (3), (4), as they involve the least amount of calculation, are to be preferred; moreover, as half an angle of a triangle is always acute, the formulæ (3), (4), (5) are never ambiguous except under peculiar circumstances. And here it well to apprise you that trigonometrical formulæ, although all correct, are not all equally suited to give correct numerical This arises from the unavoidable slight imperfections of trigonoı tables. These are, of course, computed only to a finite nuı decimal places—seldom more than seven places; hence, if two angles should differ so little from each other that their sines or agree to this extent of decimals, the tables would not enable us tinguish that difference. Now for arcs very little below and veı above 90°, it is plain, from the conformation of the circle, that tl as well as the sines of all arcs intermediate to them, differ ve from one another; so that, if any one of them is to be inferred sine, as given in the tables, it might happen from the coincidenc leading decimals, that several arcs might present equal claim regarded as *the* arc. It is always safer, therefore, when an anş near 90° is to be determined, to avoid taking the *sine* as a guid angle; a *cosine* is much better. On the contrary, when the angle small, the sine or tangent will be a better guide to it than the It thus appears that formula (3) will be unsuitable when the angle is foreseen to be very nearly 180°, and formula (4) will be un when the angle is very small. Formula (6) is open to the sam tion as (3) in the extreme case supposed, but in a less degree.

Although tangents of arcs or angles near 90° differ very rapi as the differences by which they increase are very unequal, an proportioning for seconds, we proceed on the assumption tl tangent increases uniformly through 60″, formula (5) is not so as (4) for angles near 90°.

99. Instead of a rule in words, it will be better to work by other of the formulæ (3), (4); a model of the arrangement logarithmic operation by each of which is here given :—

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

Form of operation.

Logs

a

b arith. comp.

c arith. comp.

2) sum

s

$s-b$

$s-c$

2) sum of logs

$\sin\frac{1}{2}A$

$$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$$

Form of operation.

L

a

b arith. comp.

c arith. comp.

2) sum

s

$s-a$

2) sum

$\cos\frac{1}{2}A$

No correction is to be made in the sum of the logarithms on account of the arithmetical complements, because the $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$, as determined by the formulæ, are to be increased by 10 to adapt them to the *logarithmic* values; and although *two* 10's are introduced by taking the complements as above, yet, as after the division of the sum of the logs by 2, only *one* superfluous 10 can enter, it is the 10 actually wanted to adapt the corresponding sine or cosine to the tables.

NOTE.—If the sides are small numbers, that is to say, not consisting of more than three figures each, then either of the formulæ

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{b^2+(c+a)(c-a)}{2bc}$$

$$\text{or, } \cos A = 1 - \frac{(a-b+c)(a+b-c)}{2bc} = \frac{(a+b+c)(b+c-a)}{2bc} - 1$$

may be conveniently used, the reference being, of course, to the natural cosines.

In working a particular example, you should always commence by writing down from memory the general form you intend to employ, so as to be independent of aid from the book.

(1) In a triangle ABC are given the sides, namely,

$$a = 1{\cdot}372,\ b = 6,\ c = 5{\cdot}523$$

to find the angle A.

$$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\begin{array}{lll}
a = 1{\cdot}372 & & \\
b = 6 & \text{arith. comp.} & 9{\cdot}221849 \\
c = 5{\cdot}523 & \text{arith. comp.} & 9{\cdot}257825 \\
\hline
2\,)\,12{\cdot}895 & & \\
\hline
s = 6{\cdot}4475 & & \\
s-b = \ {\cdot}4475 & \dots\dots\dots\dots & \bar{1}{\cdot}657793 \\
s-c = \ {\cdot}9245 & \dots\dots\dots\dots & \bar{1}{\cdot}965907 \\
\hline
 & & 2\,)\,18{\cdot}096374 \\
\hline
\sin\tfrac{1}{2}A,\ \ 6^\circ\ 24'\ 55'' & \dots\dots & 9{\cdot}048187 \\
\qquad\qquad 2 & & \\
\hline
\therefore\ A = 12^\circ\ 49'\ 50'' & &
\end{array}$$

Let us now work the example by the second of the formulæ in the note, namely,

$$\cos A = \frac{(a+b+c)(b+c-a)}{2bc} - 1$$

$$\begin{array}{rl} a= & 1\cdot372 \\ b= & 6 \\ c= & 5\cdot523 \\ \hline a+b+c= & 12\cdot895 \\ b+c-a= & 10\cdot151 \end{array}$$

$$\begin{array}{rrr} & & 12895 \\ & & 64475 \\ & 5\cdot523 & 12895 \\ 6\times2= & 12 & 12895 \\ \hline & 66\cdot276 &)\ 130\cdot897145\ (\ 1\cdot975003;\quad \cdot975003=\cos 12^\circ\ 49'\ 50'' \end{array}$$

$$\begin{array}{r} 66276 \\ \hline 646211 \\ 596484 \\ \hline 497274 \\ 463932 \\ \hline 33342 \\ 33138 \\ \hline 204 \\ 199 \\ \hline 5 \end{array}$$

100. When all three of the angles are to be determined, the formula (5) will, in general, be found the most convenient; for the three expressions

$$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\quad \tan\tfrac{1}{2}B = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}},$$

$$\tan\tfrac{1}{2}C = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}},$$

all involve the same factors, namely s, $(s-a)$, $(s-b)$, $(s-c)$; a circumstance of which we may take advantage; for having found $\tan\frac{1}{2}A$, we have

$$\tan\tfrac{1}{2}B = \frac{s-a}{s-b}\tan\tfrac{1}{2}A,\quad \tan\tfrac{1}{2}C = \frac{s-a}{s-c}\tan\tfrac{1}{2}A$$

$$= \tan\tfrac{1}{2}A \div \frac{s-b}{s-a},\qquad = \tan\tfrac{1}{2}A \div \frac{s-c}{s-a}$$

$$\therefore\ \log\tan\tfrac{1}{2}B=\log\tan\tfrac{1}{2}A-\log(s-b)-\text{colog}^{*}(s-a)+10$$
$$\log\tan\tfrac{1}{2}C=\log\tan\tfrac{1}{2}A-\log(s-c)-\text{colog}\ (s-a)+10$$

The following example shows how these are brought into use in the way of condensing the work.

(2) Given the sides $a=95\cdot12$, $b=162\cdot34$, $c=98$, to find all the angles A, B, C,

$$\tan\tfrac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$a=\ \ 95\cdot12$
$b=162\cdot34$
$c=\ \ 98$

$2\ )\ 355\cdot46$

$s=177\cdot73$ arith. comp. $7\cdot750239$
$s-a=\ 82\cdot61$ arith. comp. $8\cdot082967$ [p]
$s-b=\ 15\cdot39$ ………… $1\cdot187239$ [q]
$s-c=\ 79\cdot73$ ………… $1\cdot901622$ [r]

$2\ )\ 18\cdot922067$

$\tan\frac{1}{2}A$, 16° 7′ 27″ …… $9\cdot461033$ from this take [p]

$1\cdot378066$ from this take [q], [r]

$\tan\frac{1}{2}B$, 57° 12′ 4″ …… $10\cdot190827$ } 10 added
$\tan\frac{1}{2}C$, 16° 40′ 29″ …… $9\cdot476444$ }

Verification 90° 0′ 0″

We might have determined $\frac{1}{2}C$ thus, $90°-\frac{1}{2}A-\frac{1}{2}B$; but it is always better to find *all* the half angles, as above; inasmuch as we can then verify the correctness of the operations as here shown. It will, however, usually happen that, in summing up the half angles thus computed, there will be a slight departure from 90°, on account of unavoidable small errors of observation in measuring the sides, and because, moreover, of the limited extent of the decimals in the tables. The usual way of applying the correction in such a case, is to divide the difference from 90° equally among the computed angles. The foregoing example is taken, with slight modification, from Young's "Treatise on Plane and Spherical Trigonometry," to which reference may be made for several useful applications of the theory, which want of space precludes our entering upon in the present Compendium.

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* An arithmetical complement is sometimes for brevity called a *colog*.

Examples for Exercise.

(1) Given the three sides as follows, to find the angle A.

$$a=3387{\cdot}97,\ b=2065,\ c=1637$$

(2) Given $a=174{\cdot}07$, $b=232$, $c=345$, to find A, B, C.

(3) Given the sides of a triangle 4, 5, and 6 respectively; required the angles.

(4) Given $a=15{\cdot}236$, $b=12{\cdot}414$, $c=9{\cdot}018$, to find B.

(5) Three distant objects A, B, C, are at the vertices of a triangle, of which the sides are known, namely

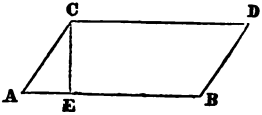
$$BC=297 \text{ feet},\ AC=328 \text{ feet},\ AB=462 \text{ feet};$$

a person in CA prolonged finds the angle subtended by AB, at the spot where he stands, to be 34° 16′ 21″; required his distance from each of the three objects A, B, C.

ON THE AREAS OF PARALLELOGRAMS, TRIANGLES, AND REGULAR POLYGONS.

Parallelogram and Triangle.

101. As every parallelogram is equal in surface or *area* to a rectangle having the same base and the same altitude, the numerical expression for the area of a parallelogram is found by multiplying the number of units of measure (whether inches, or feet, or yards, &c.) in the base, by the number of units of measure in the perpendicular or altitude. This rule is suggested by *geometry* (Euc., Prop. 35, Book i.); but sometimes it is more convenient, especially in surveying, to measure the two sides of the parallelogram, and the angle between them, rather than one side and the perpendicular upon it from the other side. In this case, the means of computing the area must be suggested by *Trigonometry*.



Let AD be a parallelogram, and CE its altitude; then, in the right-angled triangle AEC, we shall have

$$CE=AC\sin A\ ;$$

$$\therefore \text{Area}=AB\cdot CE=AB\cdot AC\sin A\,..\,(1)$$

so that the area of a parallelogram is equal to the product of any two adjacent sides multiplied by the sine of the angle between them.

The trouble of finding the perpendicular CE may thus be avoided, whenever it is more convenient, in practice, to measure the angle A. But the measurement of A even may be dispensed with by availing ourselves of the formula (6) at page 153, provided, in addition to AC, AB, we also measure the distance of C and B; for calling the sides including the angle A, b, and c, and the distance CB, a, that formula gives us

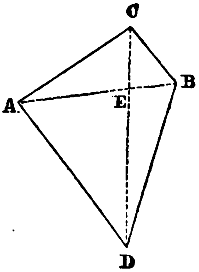
$$\sin A=\frac{2}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}}$$

and multiplying this by bc, the product of the sides, agreeably to (1) above, we have

$$\text{Area}=2\sqrt{\{s(s-a)(s-b)(s-c)\}}\ \ldots .(2).$$

And since every triangle is equal in area to half a parallelogram of the same base and altitude (Euc., Prop. 41, i.), the area of a triangle, whose sides are a, b, c, will be found by taking half of either of the expressions (1), (2) above, thus

$$\Delta=\tfrac{1}{2}bc\sin A=\sqrt{\{s(s-a)(s-b)(s-c)\}}\ldots.(3).$$



We may hence compute the area of any plane quadrilateral, however irregular, by simply measuring the two diagonals of it, and one of the angles between them.

Thus, let the annexed figure be any plane quadrilateral, the diagonals AB, CD, of which are measured, as also one of the angles at E, it matters not which, because the sine of an angle is the same as the sine of its supplement. The figure is divided, by the diagonals, into four triangles, the areas of which, by (1) above, are

$$\tfrac{1}{2}AE\cdot EC\sin E,\ \tfrac{1}{2}EB\cdot EC\sin E,\ \tfrac{1}{2}AE\cdot DE\sin E,\ \tfrac{1}{2}EB\cdot DE\sin E$$

hence the sum of all four, that is, the area of the quadrilateral, is

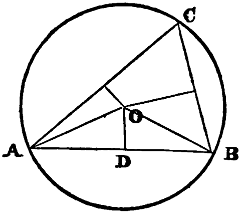
$$\left.\begin{array}{r}\tfrac{1}{2}(AE+EB)\,EC\sin E\\ +\tfrac{1}{2}(AE+EB)\,DE\sin E\end{array}\right\}=\tfrac{1}{2}(AE+EB)(DE+EC)\sin E$$

that is, $\quad \text{Area}=\tfrac{1}{2}AB\cdot DC\sin E\ldots\ldots\ldots(4)$

so that, in surveying an irregular four-sided field, if rods be set up at the four corners, and then an intermediate rod E, be so adjusted that E and C are seen in the same straight line from D, and E and B in the same straight line from A, then the angle at E, subtended by any two of the corner-rods being taken, and AB, CD measured, the area of the field may be easily computed by the formula (4); that is, by multiplying half the product of the measured diagonals by the sine of the angle between them.

It is, of course, to be understood, all along, that when we speak of multiplying *lines*, we merely mean the multiplication of the *abstract numbers* which express how many feet, yards, &c., those lines contain; the product being the *number* of square feet, yards, &c.

Circumscribed and Inscribed Circles.



102. Let ABC be any triangle; if any two of its sides be bisected by perpendiculars, these perpendiculars will meet in the centre O of the circumscribed circle (Euc., Prop. 5, Book iv.) OA will therefore be the radius of that circle; call it R; we shall then have $AD=R\sin AOD$; but AOD, half the angle AOB at the centre, is equal to the angle C of the triangle (Euc. Prop. 20, Book iii.);

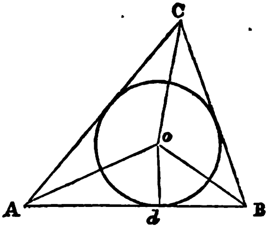
$$\therefore\ \tfrac{1}{2}c=R\sin C.\quad\text{Similarly } \tfrac{1}{2}b=R\sin B,\ \text{and } \tfrac{1}{2}a=R\sin A;$$

and from either of these expressions we get for R, by the formula (6), at page 158,

$$R=\frac{\frac{1}{4}abc}{\sqrt{\{s(s-a)(s-b)(s-c)\}}}\ldots\ldots(1)$$

But, Equation (3), page 159, the denominator of this expression is the area of the Δ,

$$\therefore\ R=\frac{abc}{4\Delta}\ \therefore\ \Delta=\frac{abc}{4R}$$



Let now o be the centre of the *inscribed* circle; then o is the intersection of the lines Ao, Bo, bisecting any two of the angles (Euc., Prop. 4, Book iv.), and the perpendicular od is the radius: call it r; we shall then have

$\frac{1}{2}cr=$ triangle oAB. Similarly
$\frac{1}{2}br=$ tri. oCA, $\frac{1}{2}ar=$ tri. oCB.

And since the sum of these triangles is the whole triangle $ABC=\frac{1}{2}bc\sin A$, we have,

$$\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}(a+b+c)r=sr\ \therefore\ r=\frac{bc}{2s}\sin A$$

$$\therefore\ (\text{Equ. 3, p. 159})\ r=\frac{\Delta}{s}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\ldots.(2)$$

Consequently, dividing (1) by (2),

$$\frac{R}{r}=\frac{\frac{1}{4}abc}{(s-a)(s-b)(s-c)}$$

If the triangle be equilateral, any side being a, then, p being the perpendicular upon it,

$$\Delta=\frac{1}{2}ap=\frac{a^3}{4R}\ \therefore\ p=\frac{a^2}{2R};\ \text{also}\ \Delta=\frac{1}{2}ap=\frac{3}{2}ar\ \therefore\ p=3r$$

that is, the perpendicular is equal to the square of a side divided by the diameter of the circumscribed circle, or to three times the radius of the inscribed circle. Also from (1), (2)

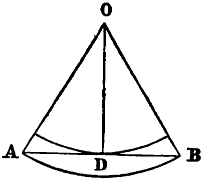
$$R=\frac{1}{3}a\sqrt{3},\ r=\frac{1}{6}a\sqrt{3}$$

so that $R=2r$, or the radius of the circumscribed circle is double that of the inscribed circle. And from either of the foregoing expressions for the equilateral Δ, we have $\Delta=\frac{1}{4}a^2\sqrt{3}$.

Regular Polygon.

103. Let a side AB of a regular polygon of n sides be represented by a, and let O be the common centre of the circumscribed and inscribed circles; the radii of these OA, OD being R, r.

The sum of all the angles at O, subtended by the n sides, is 360°, or



2π; it being customary to denote 180° by π, so that any one of these angles will be expressed by $\frac{2\pi}{n}$;

$$\therefore\ R=\frac{1}{2}a\operatorname{cosec}\frac{\pi}{n},\quad r=\frac{1}{2}a\cot\frac{\pi}{n}\ldots\ldots(1)$$

since the angle O is bisected by r. Also the area of the triangle OAB being $\frac{1}{2}ar$, if S be put for the areas of all the n triangles, that is, for the surface of the whole polygon, we shall have

$$S=\frac{1}{4}na^2\cot\frac{\pi}{n}$$

Also for a side a of the polygon, we have by (1), remembering that $\operatorname{cosec}=\frac{1}{\sin}$, and $\cot=\frac{1}{\tan}$,

$$a=2R\sin\frac{\pi}{n}=2r\tan\frac{\pi}{n}$$

Putting these values successively for a, in the expression for S, remembering that $\cot=\frac{\cos}{\sin}$, we have

$$S=nR^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}=nr^2\tan\frac{\pi}{n}$$

But, page 153, O being any angle of a triangle,

$$\sin\frac{1}{2}O\cos\frac{1}{2}O=\frac{1}{2}\sin O$$

therefore, when $O=\frac{2\pi}{n}$,

$$\sin\frac{\pi}{n}\cos\frac{\pi}{n}=\frac{1}{2}\sin\frac{2\pi}{n}$$

Hence the two expressions for the area of the polygon are

$$S=\frac{1}{2}nR^2\sin\frac{2\pi}{n}=nr^2\tan\frac{2\pi}{n}$$

104. In the foregoing investigation, the π has been employed to represent 180°, that is, two right angles without any reference to the arc which measures these degrees; but, at the commencement of the present treatise, it was seen that the numerical values of all trigonometrical quantities are estimated according to the scale of radius = 1. Every trigonometrical arc, therefore, has 1 for its radius; and it is plain that whether any such arc be expressed in angular or in linear measure, none of the deductions of trigonometry can be disturbed. It is sometimes convenient to take π—when referring to the trigonometrical semicircle—in *linear* measure; that is, to regard it as the length of a semicircle to radius 1. With this interpretation of π, we may, with propriety, speak of its *ratio* to a line, which we could not do if π stood exclusively for two right angles. Now, regarding π in this

light, we have the ratio $\tan\frac{\pi}{n} : \frac{\pi}{n}$ continually approaching nearer and nearer to a ratio of equality, that is, to 1, as n increases; that is, as the length of the arc diminishes, the ratio actually reaching 1 when n reaches infinity. If, therefore, for the preceding expression for S, we write its equivalent

$$S = \pi r^2 \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}},$$

and then suppose $n = \infty$, in which case the polygon must be confounded with the circle, and be equal in surface to it, we shall have

$$\text{Area of Circle, of rad } r, = \pi r^2.$$

Hence, the area of a circle is found by multiplying the semicircumference of a circle whose radius is 1, by the square of the radius of the proposed circle. How the circumference to radius 1 is found, will be shown in next page.

It is of importance that you should keep in mind the double sense in which the symbol π is used in analysis. In all purely trigonometrical formulæ, π may be regarded indifferently, either as 180°, that is, two right angles, or as the number 3·1416, which will presently be shown to be the numerical expression for the length of a semicircumference to radius 1*; and more generally, any angle may, in all such formulæ, be replaced by the *linear* measure of its subtending trigonometrical arc; but, when the radius is other than unity, π always means 180°, or two right angles; the linear measure of $A°$, on the trigonometrical circle, if multiplied by R, is the linear measure of $A°$ on the circle of radius R; as is obvious. The linear measure of $A°$ on this circle is therefore, length of

$$A°, \text{ to rad } R, = \frac{3·1416}{180} A \times R = \frac{·5236}{30} A \times R$$

1. Suppose, for instance, it were required to find the length of 45° of a circle whose radius is 10 feet.

$$\text{Length of } 45°, \text{ to rad } 10, = \frac{·5236}{30} \times 45 \times 10 = ·5236 \times 15 = 7·854$$

$$\therefore\ 45° = 7·854$$

2. How many degrees are there in an arc† whose length is that of the radius to which it belongs?

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[*] See the note in next page.

[†] Professor De Morgan proposes to call the degrees in an arc its *gradual* (pronounced *gra-dual*) measure; and the *length* of it its *arcual measure*, terms sufficiently brief and significant for general use.

Length of A°, to rad R, $=R=\frac{\cdot5236}{30}A\times R$

$$\therefore\quad A=\frac{30}{\cdot5236}=57\cdot295\ldots,\text{ the number of degrees}$$

$$\therefore\quad A^\circ=57^\circ\cdot295\ldots$$

In like manner, if the length of A°, to radius R, be a, then

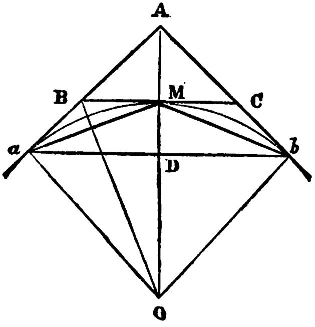
$$a=\frac{\cdot5236}{30}A\times R\ \therefore\ A=\frac{a}{R}\cdot\frac{30}{\cdot5236}\ \therefore\ A^\circ=\frac{a}{R}\times57^\circ\cdot295\ldots$$

so that if the *length* of an arc be divided by its radius, and $57^\circ\cdot295\ldots$ be multiplied by the quotient, the result will be the *degrees* in that arc. The fraction $\frac{a}{R}$ is the trigonometrical circular measure of the angle A°; for the length a, of A° in a circle of radius R, is to the length a', of A° in a circle whose radius is R', as R to R', $\therefore$ when $R'=1$, $\frac{a}{R}=a'$, the trigonometrical circular measure of the angle subtended by a; this is often called simply *the circular measure* of the angle :—it is the length of the arc, subtending the angle, divided by the radius of that arc. And the constant $57^\circ\cdot295$ (or more correctly $57^\circ\cdot2957795$)—the degrees in a length of arc equal to R—is called the *unit of circular measure*, or the unit of angular measure.

All the quantities concerned in trigonometry are thus referred to one common standard of measure, namely, the *radius*; if p be the trigonometrical sine, or cosine, or tangent, &c. of an arc, radius $\times p$ is the linear measure of it; in like manner, radius $\times 3\cdot14159\ldots$ is the linear measure of the semicircumference; and as an arc, equal to radius in length, is $57^\circ\cdot2957795\ldots$, the circular measure of the semicircumference is $57^\circ\cdot2957795\ldots\times3\cdot14159\ldots=180^\circ$.

NOTE. *On the area of a circle, and the length of its circumference.*

105. Let ab be the side of an inscribed polygon of n sides; and let the surface of it be represented by p_n; the tangents aA, bA, will each be half a side of a similar circumscribed polygon: call the surface of this polygon P_n. Let M be the middle of the arc aMb; then will the chords aM, bM, be sides of an inscribed polygon of $2n$ sides, the representation of its surface being p_{2n}, while the tangent BMC will be the side of a similar circumscribed polygon, the surface of which is represented by P_{2n}. All this is proved in Euclid's fourth book, and the following are obvious truths :—



OaD is the same part of p_n that OaA is of P_n
" " " that OaM is of p_{2n}
" " " that $OaBM$ is of P_{2n}

for each of these spaces must be repeated exactly the same number of times to complete the polygon of which it is a part; and since magnitudes are to one another as their like multiples or submultiples, whatever proportion exists among these *parts* must also exist among the *wholes*.

1. By similar right-angled triangles ODa, OaA

$$OD : Oa :: Oa : OA$$

that is, $$OD : OM :: OM : OA \ldots\ldots (1)$$

and since the altitude aD is the same for the three triangles ODa, OMa, OAa; and that triangles of the same altitude are to one another as their bases, it follows from the above proportion (1), that

$$ODa : OMa :: OMa : OAa$$

Consequently, the numerical measure of the surface of the triangle OMa is a mean proportional between the numerical measures of ODa, OAa; and therefore, *the surface of* p_{2n} *is a mean proportional between the surfaces of* p_n *and* P_n, that is, $p_{2n}=\sqrt{p_nP_n} \ldots (A)$.

2. By similar right-angled triangles ODa, BMA (the angle $BAM = OaD$, because OaA is a right angle) we have

$$OD : Oa :: BM : BA$$

that is, $$OD : OM :: aB : BA \ldots (2)$$

and since the altitude aD is the same for the two triangles ODa, OMa; and the altitude Oa the same for the two triangles OaB, OBA, and that triangles of the same altitude are to one another as their bases, it follows from the above proportion (2), that

$$ODa : OMa :: OaB : OBA$$

therefore (Euclid, 13 and 6 of v.)

$$ODa + OMa : 2ODa :: OaB + OBA : 2OaB$$

$$\therefore p_n + p_{2n} : 2p_n :: P_n : P_{2n}$$

$$\therefore P_{2n} = \frac{2p_nP_n}{p_n + p_{2n}} \ldots\ldots (B)$$

Let the radius of the circle be numerically represented by 1; and first let the inscribed and circumscribed polygons be *squares*; that is, let $n=4$; the surface of the inscribed square will be 2, that of the circumscribed square 4, that is, $p_4=2$; $P_4=4$; consequently, by (A) and (B) above,

$$p_8=\sqrt{8}=2\cdot8284271,\ P_8=\frac{2\times 8}{2+\sqrt{8}}=\frac{16}{4\cdot8284271}=3\cdot3137085$$

$$p_{16}=\sqrt{(3\cdot3137085\sqrt{8})}=3\cdot0614674,$$

$$P_{16}=\frac{2\times 3\cdot3137085\sqrt{8}}{\sqrt{8}+3\cdot0614674}=3\cdot1825979$$

And by continuing this operation, we may get, in succession, p_{32}, P_{32}; p_{64}, P_{64}; and so on to any extent, as in the following table.

Surfaces of inscribed and circumscribed Polygons, rad=1.

No. of sides.	Surf. of ins. Pol.	Surf. of circ. Pol.
4	2·0000000	4·0000000
8	2·8284271	3·3137085
16	3·0614674	3·1825979
32	3·1214451	3·1517249
64	3·1365485	3·1441184
128	3·1403311	3·1422236
256	3·1412772	3·1417504
512	3·1415138	3·1416321
1024	3·1415729	3·1416025
2048	3·1415877	3·1415951
4096	3·1415914	3·1415933
8192	3·1415923	3·1415928
16384	3·1415925	3·1415927
32768	3·1415926	3·1415926

106. We thus see that the numerical expressions for the surfaces of the inscribed and circumscribed polygons of 32768 sides, agree as far as seven places of decimals; and, consequently, as the surface of the circle itself is always intermediate in value between the surfaces of the corresponding pair of polygons, the numerical expression for the area of a circle, of radius 1, as far as seven places of decimals, must be 3·1415926. Now it was shown at page 162, that

$$\text{Area of Circle, of rad } r, = \pi r^2$$

in which π is the numerical expression for the semicircumference of a circle to radius 1, therefore,

$$\text{Area of Circle, of rad } 1, = \pi$$

that is, the numerical value of π, is the same as the numerical value of the area; the only difference being that the former expresses linear measure, and the latter square measure,

$$\therefore\quad \pi = 3{\cdot}1415926^*$$

which, to four places of decimals, an extent sufficient for common purposes, is 3·1416, the semicircumference of a circle whose radius is 1; or the whole circumference of a circle whose diameter is 1, because circumferences of circles are to one another as their radii.

PART II.

THE TRIGONOMETRICAL ANALYSIS.

107. The foregoing part of the present treatise has been devoted to Trigonometry proper; the angles concerned being all comprised within the limits of 0° and 180°, agreeably to the Euclidean definition of an

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* That this number is an interminable decimal, will be proved, in a way which the author conceives to be original, at the end of the present volume

angle. There is no necessity to go beyond these limitations in order to establish all the relations which exist between the sides and angles of a triangle, and for this single purpose the foregoing investigations are amply sufficient, and may be regarded as complete in themselves. But for more advanced applications, a more extended theory of angular magnitude must be provided; and to give a compendious view of this more general theory is the object of this SECOND PART.

The trigonometrical analysis, as the name indicates, establishes the general properties of angular magnitude by means of algebraic symbols. Geometrical diagrams are employed only at the outset, and that merely for the purpose of explaining terms, and of settling certain fundamental principles and conventions; the subject is then turned over to the generalizations of algebra, and all its conclusions exhibited in analytical formulæ.

Previously, however, to entering upon the main business of the present section, it may be well, by way of introduction, to show you the comprehensive character of algebraical processes, by applying them to develope the trigonometrical theory of the plane triangle, so fully discussed, by a union of algebra and geometry, in the preceding section.

Analysis of the Triangle.

108. Referring to the diagrams at page 146, we have $c=a\cos B+b\cos A$; and, similarly, for the sides a and b; that is,

$$a=b\cos C+c\cos B$$
$$b=c\cos A+a\cos B$$
$$c=a\cos B+b\cos A$$

Here are three equations, from which, if any three of the six quantities entering them be given—except the three angles—the remaining three may be determined, from the algebra of simultaneous simple equations. If the sides a, b, c, were taken m times as great, that is, if we were to multiply each equation by m, then, instead of a, b, c, we should merely have to write ma, mb, mc; and, removing the factor m by division, we should return to the equations above; so that the values of $\cos A$, $\cos B$, $\cos C$, would be the same for the triangle whose sides are ma, mb, mc, as for that whose sides are a, b, c; this shows that the sides are not determinable from the angles. It teaches us, moreover, that the algebraical expressions for the cosines, deduced from the above equations, have all the terms, in both numerators and denominators, of *the same dimensions;* inasmuch as the expressions would remain unaltered in value though ma, mb, mc, were to be written in them for a, b, c.

The three equations above, as they involve all the six parts of a triangle, and are solvable whichever three of those parts (except the sides) are unknown, evidently involve the whole theory of plane trigonometry as far as the triangle is concerned; nevertheless, as the formulæ of solution, derived from these equations, would not present themselves in a form fitted for the immediate application of logarithms, certain transformations of them would be necessary to give to them the desired practical value. The invention of logarithms, therefore, however much it may facilitate calculation, and ease the labour of the computor, has certainly complicated the task of the theorist.

In order to obtain expressions for the cosines from the preceding equations, multiply them successively by a, b, c; we thus have

$$a^2 = ab\cos C + ac\cos B$$
$$b^2 = bc\cos A + ab\cos C$$
$$c^2 = ac\cos B + bc\cos A$$

and subtracting each of these from the sum of the other two, there results

$$b^2 + c^2 - a^2 = 2bc\cos A \quad \therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$a^2 + c^2 - b^2 = 2ac\cos B \quad \therefore \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$a^2 + b^2 - c^2 = 2ab\cos C \quad \therefore \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

as otherwise determined at page 152; and we see that all the terms concerned are of the same dimensions, namely, two dimensions; the right-hand member of each equation being an abstract number, as of course it ought to be, since the cosines themselves are abstract numbers. If $a = b = c$, then $\cos A = \cos 60° = \frac{1}{2}$.

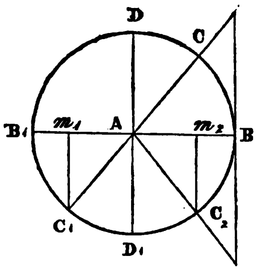
We shall now proceed to the more special object of the present section.

On Angular Magnitude in general, and the geometrical lines or ratios connected with it.

109. Angular magnitude may be considered as generated by the revolution of a straight line AC starting from a fixed position AB.

There is no necessary limit to the extent of this magnitude; upon completing one revolution the angular magnitude generated will amount to 360°, and by still continuing the rotation, the angular magnitude may reach any number of times 360°. In this way we may, with propriety, speak of arcs or angles (angular magnitude) of 200°, 380°, 740°, &c., such angular magnitudes being, each, nothing but the *sum* of a certain number of the angles defined by Euclid. And whenever we speak of an angle greater than 180°, we are to be understood in this sense:—we speak of the *sum* of a set of Euclid's angles, as *an* angle. In fact, even Euclid himself, in one part of his performance, gives countenance to this extension of the meaning of the term *angle:* in the demonstration of Prop. 33 of Book vi., he regards any "multiple of the angle BGC" as *an angle.*

110. We have now, therefore, to consider the trigonometrical lines or ratios, already defined, in connection with angles, and their measuring arcs, when these exceed 180°. And first we have to remark, that in the application of algebra to any geometrical inquiry in which the *position*, as well as the *lengths* of the lines concerned, is to be attended to, an algebraic sign of *direction*, as well as an algebraic symbol of



length, is always introduced. If algebra did not supply the means for distinguishing directions as well as lengths, it could not fully perform the office of diagrams addressed to the eye, and could therefore but imperfectly replace geometrical constructions.

As already noticed, the directive signs supplied by algebra are + and −, the marks employed at the beginning of algebra to indicate the opposites, addition and subtraction. But a student has to go but a very little way in that science before he perceives that these signs—of their own accord—if we may so express it, emancipate themselves from this restriction, and offer themselves to indicate *opposites* of any kind. You have seen many instances in which, upon your marking *gain* as +, algebra has marked *loss* as −, without your having formally *agreed* to do so. In like manner, if *altitude* were marked +, you could not avoid *depression* being marked − ; and you were thus brought to see that the signs + and −, employed by you at the outset for a special individual purpose, were, in fact, the general algebraic indications of *opposites* or *contraries*. It is, of course, matter of choice *which* of two opposites shall be marked +, but that the other shall be marked −, is matter, not of choice, but of *necessity*.

In the present subject, the line AB, from which the revolving radius starts, is always marked +, the opposite line AB' is therefore −. Again, AD is marked +, and therefore AD', − ; and all parallels to these lines, as they have the same directions, are marked in the same way. Thus, in the foregoing diagram, every parallel to AB, drawn from any point in DD', and to the *right* of DD', is + ; and every parallel drawn to the *left* is −. In like manner, every parallel to AD, drawn from any point in BB', *above* BB', is +, and *below* BB', −.

The same directive marks are attached to the *arcs*, which are all considered to originate at B, and to be + when generated in the direction BCD, &c. ; they are therefore −, when generated in the contrary direction BC_2D', &c. ; and what applies to the arcs, applies, of course, to the angles they subtend or measure.

These principles being admitted, the sine of the arc BDC_1, or of the angle subtended, is, by definition, C_1m_1 ; the numerical value of the radius AB being 1, if the numerical value of C_1m_1 be p, then $\sin BDC_1 = -p$; also if the numerical value of the cosine Am_1 be q, then $\cos BDC_1 = -q$. In like manner $\sin BDC_1C_2 = -p'$, and $\cos BDC_1C_2 = +q'$, or q', the plus sign being always suppressed though understood. You see that the positive arc (or angle) BDC_1 has the same sign and cosine as the negative arc (or angle) $BD'C_1$; and the positive arc (or angle) BDC_1C_2, the same sine and cosine as the negative arc (or angle) BC_2. And a glance at the diagram is sufficient to show that the same remark applies to the tangent, which it is easy to see is always $=\frac{\sin}{\cos}$. Indeed, from sine and cosine, all the other trigonometrical lines are derivable, both as to numerical value and sign, as in the more limited consideration of these lines at page 135. This you can easily prove for yourself, without there being any occasion here to complicate the diagram by exhibiting all these lines in different positions. But it is necessary to make a special remark or two in reference to the secant.

111. The secant is a line of a peculiar character; there is only one pair of positions of the secant, which are the opposites of one another, as already noticed at page 135; and although secant is $=\frac{1}{\cos}$, yet it is proper that you should distinctly see that this algebraical equivalence is in perfect accordance with the geometrical conditions, and with the general principle of the directive signs already laid down, but laid down, be it remembered, only in reference to lines parallel to either DD' or BB'. There seems to be nothing in this principle to justify our attributing *any* sign to a line oblique to either of these, and radiating from their intersection A.

To settle the point, your attention must be directed to a general mathematical truth, which cannot but be regarded as an axiom: it is this—

Any quantity which continuously diminishes in magnitude must reach the value 0 *before it can change its sign.*

We cannot conceive of a positive quantity passing from + to −, by continuous diminution of value, without passing through *zero*; such is evidently the case with the sine, cosine, tangent, &c. Let us watch the varying progress of the secant. At the commencement of its existence, the secant coincides with AB, it is then the secant of 0°; and conformably to the directive principle, as applied to AB, it is +. In revolving towards AD, the secant extends itself up to the tangent, continuously increasing in length, till, arriving at the position AD, it becomes infinite, and finishes its course; throughout the range, in virtue of the above axiom, it is positive:—it cannot have changed the sign with which it started.

AD, thus extended to infinity, does not *continue* to revolve to complete the cycle of secants; it is the *opposite* line AD', extended to infinity, that supplies the completion of this cycle, as is obvious; this opposite line is −; and, throughout its entire progress towards AB, as it does not pass through zero, it must continue −, and be − even when it arrives at coincidence with AB; that is, $\sec 180° = -1$; and this quite agrees with the algebraical condition $\sec = \frac{1}{\cos}$. It is thus strictly in accordance with the above axiomatic principle, and at the same time with the sign-convention, as to direction, that the radiating line called *secant*—which is ever varying in position—should start from AB with a positive sign, and return to it with a negative sign; the explanation of the apparent anomaly being that the radiating line has been changed on the journey; the line that *arrives* is *not* the line that *started*. Attention to the axiomatic principle above, and to the fact that the secant changes sign, by becoming a different line, when it arrives at D, is sufficient to enable us to fix upon the proper sign of the secant for any proposed arc or angle, and to show the universality of the relation $\sec=\frac{\sin}{\cos}$.

112. As all the other trigonometrical lines are thus deducible, as well in sign as in numerical value, from the sine and cosine, it will be sufficient to attend to the changes which these latter undergo; they are as follows:—

SINE.—Of any arc terminating in the 1st, 3rd, 5th, &c. *semicircle*, the sign is + ; for other terminations it is −.

COSINE.—Of any arc terminating in the 1st, 4th; 5th, 8th; 9th, 11th; &c. *quadrants*, the sign is + ; and of any arc terminating in the intermediate quadrants, that is, in the 2nd, 3rd; 6th, 7th; 10th, 11th; &c. the sign is −. And the *numerical value* of either of these sines or cosines is always equal to the numerical value of the sine or cosine of that arc which must be added to or subtracted from the given one, to make up a complete set of *semicircles*.

For example: 572° terminates in the 4th semicircle and 7th quadrant, because $180°\times 3=540°$; its sine is therefore −, and its cosine also −; to get three complete semicircles, 32° must be subtracted; hence

$$\sin 572°=-\sin 32°,\ \cos 572°=-\cos 32°$$

Again, take 283°; this terminates in the second semicircle, and 4th quadrant; also 77° must be added to make two complete semicircles; hence the sine is −, and the cosine +

$$\therefore\ \sin 283°=-\sin 77°,\ \cos 283°=\cos 77°$$

These principles, expressed analytically, are embodied in the following general formulæ, namely:—

$$\sin(2n\pi\pm A)=\sin[(2n+1)\pi\mp A]=\pm\sin A$$
$$\cos(2n\pi\pm A)=-\cos[(2n+1)\pi\mp A]=\cos A$$

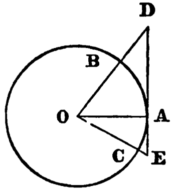
where n is any whole number, and $\pi=180°$.

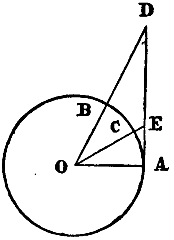
But, by mentally tracing the arc, and discovering its termination, you will always be able to determine its sine and cosine without any reference to these rules. And, in thus tracing the arc to its termination, you may, of course, take account only of that portion of it which exceeds 360°, or a multiple of 360°; any such multiple may also be added on at pleasure; for in either case the *termination* is undisturbed.

Fundamental Equations of Analytical Trigonometry:—sine and cosine of the sum and difference of two angles.

113. It has been seen that wherever an arc, originating at A, terminates, the tangent and secant of that arc always form, with the radius, a right-angled triangle situated to the right of the centre O; and moreover, that, as far as their numerical values are concerned, the sine, cosine, &c. of the Euclidean angle O, are the same as the sine, cosine, &c. of the arc; and the proper directive signs are determinable, as shown above, from the known termination of the arc. As far, therefore, as the numerical values of the trigonometrical lines are concerned, the arc AB may be substituted for the arc, whatever be its extent, of

which the tangent is AD, and the secant OD. Now let A, B, represent two arcs of any magnitude, the tangents of which are equal to the lines AD, AE, and therefore their secants to OD, OE. As far as the numerical values of the trigonometrical lines are concerned, these arcs may be replaced by AB, AC; and their sum or difference by BC.





Now, page 152, and Euclid, Props. 4 and 7, Book ii., we have

$$DE^2 = OD^2 + OE^2 - 2OE \cdot OD \cos DOE$$

$$DE^2 = AD^2 + AE^2 \pm 2AD \cdot AE$$

Hence, taking radius $=1$, and subtracting the second equation from the first,

$$0 = 1 + 1 \mp 2\tan A \tan B - 2\sec A \sec B \cos(A \pm B)$$

$$\therefore \cos(A \pm B) = \frac{1 \mp \tan A \tan B}{\sec A \sec B} \ldots\ldots (1)$$

$$= \cos A \cos B \mp \sin A \sin B$$

Again: $OD \cdot OE \sin DOE = \text{twice } \Delta DOE = OA \cdot DE$, where $OA = 1$

$$\therefore \sin(A \pm B) = \frac{\tan A \pm \tan B}{\sec A \sec B} \ldots\ldots\ldots (2)$$

$$= \sin A \cos B \pm \cos A \sin B$$

And dividing (2) by (1),

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

We have thus got expressions for the sine and cosine of the sum, and for the sine and cosine of the difference of any two arcs or angles A, B; and moreover, we have deduced, incidentally, expressions likewise for the *tangent* of the sum and difference.

The preceding equations, remember, are perfect *identities;* they imply no *condition* as to the arcs or angles A and B, which may be any whatever; and therefore nothing is assumed as to the algebraic prefix of the sines and cosines of these angles. Every numerical value, subject to variation, takes a change of sign only upon its passage through zero or infinity; nor even then, unless the linear representative of the numerical quantity changes its direction. The above expressions, therefore, are open to any interpretation, as to the values of the symbols, consistently with this universal algebraic principle. In all general investigations respecting varying quantities, subject to change of sign, we invariably avoid all assumption as to sign; and regard the symbol of quantity to comprehend numerical value, sign and all; and when an identity is arrived at, the signs, adapted to special cases, may be safely introduced; the binomial and logarithmic series furnish abundant instances of this; an algebraic identity being obtained, we modify the signs of our varying quantities at pleasure.

In the first of the preceding diagrams AC is united to AB, so as to present a continuous arc for the sum of the two; and moreover that, in all cases, the angle BOC, in one diagram or the other, may be really that due to the sum or difference of the proposed arcs, as respects the numerical values of the trigonometrical lines.

114. The expressions just deduced form the bases of all the investigations of the trigonometrical analysis; we shall repeat them in detail here, for the purpose of reference.

$$\left.\begin{aligned}\sin(A+B)&=\sin A\cos B+\cos A\sin B\\ \sin(A-B)&=\sin A\cos B-\cos A\sin B\\ \cos(A+B)&=\cos A\cos B-\sin A\sin B\\ \cos(A-B)&=\cos A\cos B+\sin A\sin B\end{aligned}\right\}\ldots.(\text{I})$$

And, in addition to these, we have also found that

$$\left.\begin{aligned}\tan(A+B)&=\frac{\tan A+\tan B}{1-\tan A\tan B}\\ \tan(A-B)&=\frac{\tan A-\tan B}{1+\tan A\tan B}\end{aligned}\right\}\ldots\ldots(\text{II})$$

If the first only of the formulæ (I) had been determined in all its generality, the other three might have been derived from it, algebraically, thus:—Put $-B$ for B, then we have

$$\sin(A-B)=\sin A\cos(-B)+\cos A\sin(-B)$$

but $\cos(-B)=\cos B$, and $\sin(-B)=-\sin B$,

$$\therefore\ \sin(A-B)=\sin A\cos B-\cos A\sin B$$

Also $\sin[90°-(A+B)]=\cos(A+B)$; and $90°-(A+B)$ is the same as $(90°-A)-B$; therefore substituting the difference $90°-(A+B)$ for the difference $A-B$, in the first member of the formulæ just deduced, and the equivalent $(90°-A)-B$ in the second member, we have

$$\begin{aligned}\cos(A+B)&=\sin(90°-A)\cos B-\cos(90°-A)\sin B\\ &=\cos A\cos B-\sin A\sin B\end{aligned}$$

and putting $-B$ for B

$$\begin{aligned}\cos(A-B)&=\cos A\cos(-B)-\sin A\sin(-B)\\ &=\cos A\cos B+\sin A\sin B\end{aligned}$$

It is satisfactory to see that we may thus change the sign of any angle entering these formulæ; for, as all others may be derived from them, you can feel no hesitation in introducing a like change in any result hereafter established; such supplementary evidence of the perfect identity of the above forms was not, however, needed to establish the fact. If we take the reciprocal of each member of (II), we have corresponding expressions for the cotangents of the sum and difference of the two angles, namely,

$$\left.\begin{aligned}\cot(A+B)&=\frac{1-\tan A\tan B}{\tan A+\tan B}=\frac{\cot A\cot B-1}{\cot B+\cot A}\\ \cot(A-B)&=\frac{1+\tan A\tan B}{\tan A-\tan B}=\frac{\cot A\cot B+1}{\cot B-\cot A}\end{aligned}\right\}\ldots.(\text{III})$$

the last forms being got by dividing numerator and denominator of the preceding by $\tan A \tan B$.

115. We stop here to notice an interesting property of the three angles of a triangle. Let A, B, C be the three angles, then since C is the supplement of $A+B$, $\tan(A+B)=-\tan C$, hence (II)

$$-\tan C=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\therefore \tan A\tan B\tan C=\tan A+\tan B+\tan C$$

that is, the *product* of the tangents of the angles is equal to the *sum*. If the three angles, instead of making 180°, make 90°, then, since C is the complement of $A+B$, $\tan(A+B)=\cot C$; hence, taking the reciprocals of the first and third members of (III), we have

$$\cot C=\frac{\cot B+\cot A}{\cot A\cot B-1}$$

$$\therefore \cot A\cot B\cot C=\cot A+\cot B+\cot C$$

The former of these curious properties holds whenever the three angles amount to an even multiple of 90°, and the latter when they amount to an odd multiple of 90°.

It must be all along remembered that it is the *numerical* values of the trigonometrical lines, to rad $=1$, with which we are dealing; our formulæ may always be generalized so as to hold, geometrically, for any length R of the radius, by writing $\frac{\sin}{R}$, $\frac{\cos}{R}$, &c. for sin, cos, &c.; thus, the first of the above properties is

$$\frac{\tan A}{R}\cdot\frac{\tan B}{R}\cdot\frac{\tan C}{R}=\frac{\tan A}{R}+\frac{\tan B}{R}+\frac{\tan C}{R}\text{ (See page 138.)}$$

116. Returning now to the formulæ (I), we have, by addition and subtraction,

$$\left.\begin{aligned}\sin(A+B)+\sin(A-B)&=\ \ 2\sin A\cos B\\ \sin(A+B)-\sin(A-B)&=\ \ 2\cos A\sin B\\ \cos(A+B)+\cos(A-B)&=\ \ 2\cos A\cos B\\ \cos(A+B)-\cos(A-B)&=-2\sin A\sin B\\ \text{or }\cos(A-B)-\cos(A+B)&=\ \ 2\sin A\sin B\end{aligned}\right\}\ \ldots.(\text{IV})$$

If $A=60°$, then $\cos A=\frac{1}{2}$ (p. 167), and therefore $2\cos A=1$; therefore, from the second of these equations, we learn that

$$\sin(60°+B)=\sin(60°-B)+\sin B$$

a formula which may evidently be turned to account in the construction of tables, or in verifying those already constructed, thus,

$$\sin 60°\ 1'=\sin 59°\ 59'+\sin 1'$$
$$\sin 60°\ 2'=\sin 59°\ 58'+\sin 2'$$
$$\sin 60°\ 3'=\sin 59°\ 57'+\sin 3'$$

&c. &c.

therefore if the sines are computed up to 60°, the remainder of the table may be supplied by the simple addition of sines previously found.

117. Going back to the fundamental formulæ (I), we have, by multiplication,

$$\sin(A+B)\sin(A-B)=\sin^2A\cos^2B-\cos^2A\sin^2B$$
$$\cos(A+B)\cos(A-B)=\cos^2A\cos^2B-\sin^2A\sin^2B$$

But $\sin^2A+\cos^2A=1$, $\sin^2B+\cos^2B=1$,

$$\therefore \sin(A+B)\sin(A-B)=\sin^2A-\sin^2B=\cos^2B-\cos^2A$$
$$\cos(A+B)\cos(A-B)=\cos^2A-\sin^2B=\cos^2B-\sin^2A$$

By putting A for B in the first of (IV), and in the expression just obtained, we have

$$\sin 2A=2\sin A\cos A,\quad \cos 2A=\cos^2A-\sin^2A$$

or, since $\cos^2A=1-\sin^2A$, and $\sin^2A=1-\cos^2A$, the last expression may be written

$$\cos 2A=1-2\sin^2A=2\cos^2A-1$$

And in like manner, putting A for B in (II),

$$\tan 2A=\frac{2\tan A}{1-\tan^2A}\ldots\ldots(\text{V})$$

These expressions for the sine, cosine, and tangent of a double arc, may of course be readily changed into those for a single arc, by putting $\frac{1}{2}A$ for A; we thus have

$$\left.\begin{array}{l}\sin A=2\sin\frac{1}{2}A\cos\frac{1}{2}A,\\ \cos A=\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A=1-2\sin^2\frac{1}{2}A=2\cos^2\frac{1}{2}A-1\end{array}\right\}\ldots\ldots(\text{VI})$$

$$\tan A=\frac{2\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}\ldots\ldots(\text{VII})$$

118. If the first of (VI) be added to, and substracted from $1=\sin^2\frac{1}{2}A+\cos^2\frac{1}{2}A$, the second member of the result will be a complete square; and we shall have

$$\left.\begin{array}{l}1+\sin A=(\sin\frac{1}{2}A+\cos\frac{1}{2}A)^2\\ 1-\sin A=(\sin\frac{1}{2}A-\cos\frac{1}{2}A)^2\end{array}\right\}\ldots\ldots(\text{VIII})$$

And we further see, from the second of (VI), that

$$1-\cos A=2\sin^2\tfrac{1}{2}A,\ \text{and}\ 1+\cos A=2\cos^2\tfrac{1}{2}A$$

two properties which, if applied to the cosines at p. 167, will convert the expressions on the right into forms fit for logarithms, as at p. 153.

Combining these with (VI), by division, we have

$$\frac{1-\cos A}{\sin A}=\tan\tfrac{1}{2}A=\frac{\sin A}{1+\cos A}=\sqrt{\frac{1-\cos A}{1+\cos A}}$$

If instead of putting $\frac{1}{2}A$ for A, we had put $\frac{1}{2n}A$, we should have got

$$\left.\begin{array}{l}\sin\frac{1}{n}A=2\sin\frac{1}{2n}A\cos\frac{1}{2n}A\\ \cos\frac{1}{n}A=1-2\sin^2\frac{1}{2n}A=2\cos^2\frac{1}{2n}A-1\end{array}\right\}\ldots\ldots(\text{IX})$$

119. If in the first and third of (IV) we put $(n-1)A$ for A, and A *for* B, there results

$$\sin nA+\sin(n-2)A=2\sin(n-1)A\cos A\ldots.(1)$$
$$\cos nA+\cos(n-2)A=2\cos(n-1)A\cos A\ldots.(2)$$

which general formulæ will furnish expressions for the sines and cosines of multiple arcs; thus, from (1),

$$\begin{aligned}
\text{putting}\quad n=2,\ \therefore\ \sin 2A&=2\sin A\cos A\\
n=3,\ \therefore\ \sin 3A&=2\sin 2A\cos A-\sin A\\
&=4\sin A\cos^2A-\sin A\\
&=4\sin A\,(1-\sin^2A)-\sin A\\
&=3\sin A-4\sin^3A\\
\&\text{c.}\qquad&\qquad\&\text{c.}
\end{aligned}$$

In like manner from (2)

$$\begin{aligned}
\text{putting}\quad n=2,\ \therefore\ \cos 2A&=2\cos^2A-1\\
n=3,\ \therefore\ \cos 3A&=2\cos 2A\cos A-\cos A\\
&=4\cos^3A-3\cos A\\
\&\text{c.}\qquad&\qquad\&\text{c.}
\end{aligned}$$

120. In the same formulæ (IV) put $\frac{1}{2}(A+B)$ for A, and $\frac{1}{2}(A-B)$ for B, then the sum of the two angles will be A, and their difference B; and (IV) will become

$$\begin{aligned}
\sin A+\sin B&=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\\
\sin A-\sin B&=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)\\
\cos A+\cos B&=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)\\
\cos B-\cos A&=2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)
\end{aligned}$$

and from these we get, by division,

$$\left.\begin{aligned}
\frac{\sin A+\sin B}{\cos A+\cos B}&=\tan\tfrac{1}{2}(A+B)\\
\frac{\sin A-\sin B}{\cos A+\cos B}&=\tan\tfrac{1}{2}(A-B)
\end{aligned}\right\}\ \therefore\ \frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

It may be interesting to notice here that this last expression furnishes the rule, given in the former part, for solving the second case of oblique triangles; thus,

$$\text{since}\quad \frac{a}{b}=\frac{\sin A}{\sin B}\ \therefore\ (\text{Alg. p. }45)\ \frac{a+b}{a-b}=\frac{\sin A+\sin B}{\sin A-\sin B}$$
$$\therefore\ \frac{a+b}{a-b}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$$

which is the rule alluded to. (See page 150.)

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121. It is easy to see, from what has now been done, that the deduction of formulæ, from the fertile source marked (I), may be continued to any extent. Having, by aid of a few geometrical considerations, established the general principles embodied in the equations (I), the investigation of the various theorems, which lie concealed under those equations, becomes the business of pure algebra. As soon as the trigonometrical lines are translated into algebraic symbols, and that the

directions of these lines are found to be as distinctly indicated by algebraic signs as by a geometrical diagram, algebra is at once recognized as competent to supply the place of geometry; and we thenceforward trust ourselves implicitly to its guidance. There is, however, a generality about algebraic results—to whatever inquiry the symbols may relate—that sometimes extends beyond the range of that inquiry; and, in order to limit them to the case in hand, they must be pruned of their exuberance, and restricted to their proper bounds, by the overruling conditions of the particular problem to which the general symbols are applied. For instance: a problem may be such as to be of impossible solution except for *positive* values of the quantity sought; yet, when submitted to the general operations of algebra, *negative* values may be among the results; the character of the problem must then be referred to, in order to point out to us the rejective values. Again, the problem may be such that, for certain cases really comprehended in it, the admissible results must, from the nature of its overruling conditions, be exclusively *positive,* while for other of its cases, the results must be exclusively *negative;* this special distinction of cases cannot, in general, be worked into the algebra, and thus be made to exercise a control over its determinations: it will furnish neither the one description of results nor the other, *exclusively,* but *both;* and it is then a matter of after consideration, *which* agrees with one set of conditions, and which with the other. Instances of this are continually presenting themselves, even from a very early stage of common algebra. If a radical sign enter into a result, we are offered a choice of interpretations of it; it may happen that *either* interpretation may be in harmony with the prescribed conditions, or *one* interpretation only may be admissible: algebra will never give us too little, though it will often supply too much.

122. The formula (VIII), page 174, give

$$\cos A + \sin A = \pm\sqrt{(1+\sin 2A)}$$

$$\cos A - \sin A = \pm\sqrt{(1-\sin 2A)}$$

and $\therefore \cos A = \pm\frac{1}{2}\{\sqrt{(1+\sin 2A)} \pm \sqrt{(1-\sin 2A)}\}$

$$\sin A = \pm\tfrac{1}{2}\{\sqrt{(1+\sin 2A)} \mp \sqrt{(1-\sin 2A)}\}$$

The right-hand side of each of these equations is either + or −, indifferently; but the left-hand side, for a given value of A, is exclusively one or the other: and which of the two is to be retained is only determinable by inquiring into the magnitude of A; should there be any difficulty in settling the matter at once, as respects the first two equations, by simple inspection, the difficulty will be removed by writing $\sin(90°-A)$ for $\cos A$, and observing the terminations of the arcs $(90°-A)$ and A. It is useful, too, to remember that, when A is less than 45°, $\cos A - \sin A$ must be +; and that when A is greater than 45° and less than 90°, it is −; and the same is of course true whatever be the length of arc, provided it terminates as here supposed; and by observing on which side the point of bisection of each quadrant the arc terminates, the sign in each case may be easily settled.

123. We shall now exhibit a few practical applications of the foregoing formulæ, and shall then terminate the present portion of the

subject with some further deductions from them in the shape of Exercises in Analytical Trigonometry.

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1. To find sin 30° and cos 30°.

If a perpendicular be drawn from the vertex C of an equilateral triangle, it will bisect both the angle C and the base c; and as each angle of an equilateral triangle is 60°, we shall have $b\cos A=\frac{1}{2}c$; that is,

$$b\cos 60^\circ=\tfrac{1}{2}b \quad \therefore \quad \cos 60^\circ=\sin 30^\circ=\tfrac{1}{2}$$

also $$\sin 60^\circ=\cos 30^\circ=\sqrt{(1-\tfrac{1}{4})}=\tfrac{1}{2}\sqrt{3}$$

2. To find sin 45° and cos 45°.

As $\sin 45^\circ=\cos 45^\circ$; and that $\sin^2 A+\cos^2 A=1$

$$\therefore \quad 2\sin^2 45^\circ=2\cos^2 45^\circ=1$$

$$\therefore \quad \sin 45^\circ=\tfrac{1}{2}\sqrt{2}, \quad \cos 45^\circ=\tfrac{1}{2}\sqrt{2}$$

3. To find sin 15° and cos 15°.

By (IV) $\cos(30^\circ+15^\circ)+\cos(30^\circ-15^\circ)=2\cos 30^\circ\cos 15^\circ$

$$\therefore \quad \cos 45^\circ=(2\cos 30^\circ-1)\cos 15^\circ$$

or $$\tfrac{1}{2}\sqrt{2}=(\sqrt{3}-1)\cos 15^\circ$$

$$\therefore \quad \cos 15^\circ=\frac{\sqrt{2}}{2(\sqrt{3}-1)}$$

Again (VI) since $\sin A\cos A=\frac{1}{2}\sin 2A$

$$\therefore \quad \sin 15^\circ\cos 15^\circ=\tfrac{1}{2}\sin 30^\circ=\tfrac{1}{4}$$

$$\therefore \sin 15^\circ=\frac{1}{4\cos 15^\circ}=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

4. To find sin 18° and cos 18°.

By (VI) $\sin 36^\circ=2\sin 18^\circ\cos 18^\circ=\cos 54^\circ$

But $54^\circ=3\times 18^\circ$, and the cosine of this multiple of 18° by p. 175 is $\cos 54^\circ$, that is, $2\sin 18^\circ\cos 18^\circ=4\cos^3 18^\circ-3\cos 18^\circ$

$$\therefore \quad 2\sin 18^\circ=4\cos^2 18^\circ-3=1-4\sin^2 18^\circ$$

$$\therefore \quad 4\sin^2 18^\circ+2\sin 18^\circ=1$$

and solving this quadratic equation, we have

$$\sin 18^\circ=\frac{-1+\sqrt{5}}{4}$$

Again, as just shown,

$$2\sin 18^\circ=4\cos^2 18^\circ-3 \quad \therefore \quad \frac{-1+\sqrt{5}}{2}+3=4\cos^2 18^\circ$$

$$\therefore \quad \cos 18^\circ=\sqrt{\frac{10+2\sqrt{5}}{4\cdot 4}}=\tfrac{1}{4}\sqrt{(10+2\sqrt{5})}$$

Hence sin 36°, sin 14°, sin 12°, and many others become known; thus

$$\sin 36^\circ=2\sin 18^\circ\cos 18^\circ=\tfrac{1}{4}\sqrt{(10-2\sqrt{5})}$$
$$\sin 14^\circ=\sin(36^\circ-18^\circ)$$
$$\sin 12^\circ=\sin(30^\circ-18^\circ) \qquad \text{(See formulæ I.)}$$
$$\sin 9^\circ=\sin(45^\circ-36^\circ)$$
$$\sin 6^\circ=\sin(36^\circ-30^\circ)$$

&c. &c.

5. Given $\sec A = 4 \sin A$, to find A.

$$\text{Since } \sec A = \frac{1}{\cos A} \quad \therefore \quad 4 \sin A \cos A = 1$$

$$\text{but } 2 \sin A \cos A = \sin 2A \quad \therefore \quad 2 \sin 2A = 1$$

$$\therefore \sin 2A = \tfrac{1}{2} \quad \therefore \text{ (ex. 1) } 2A = 30^\circ \quad \therefore A = 15^\circ$$

6. Given $\operatorname{cosec} A = \sin A + \tan \frac{1}{2} A$ to find $\sin A$.

$$\frac{1}{\sin A} = \sin A + \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A}$$

$$\frac{1}{2 \sin \frac{1}{2} A \cos \frac{1}{2} A} = \sin A + \frac{\sin \frac{1}{2} A}{\cos \frac{1}{2} A}$$

Multiplying the terms of the last fraction by $2 \sin \frac{1}{2} A$, and transposing,

$$\frac{1 - 2 \sin^2 \frac{1}{2} A}{2 \sin \frac{1}{2} A \cos \frac{1}{2} A} = \sin A$$

that is (VI)

$$\frac{\cos A}{\sin A} = \sin A$$

$$\therefore \cos A = \sin^2 A = 1 - \cos^2 A$$

$$\therefore \cos^2 A + \cos A = 1 \quad \therefore \cos A = \frac{-1 \pm \sqrt{5}}{2}$$

$$\therefore \sin A = \sqrt{\cos A} = \tfrac{1}{2} \sqrt{(-2 \pm 2\sqrt{5})}$$

Examples for Exercise.

1. Find $\sin 9^\circ$ and $\cos 9^\circ$.
2. Find $\sin 27^\circ$ and $\cos 27^\circ$.
3. Given $\sin 2x = \sin^2 3x$ to find x.
4. Prove that $\tan (45^\circ + A) - \tan (45^\circ - A) = 2 \tan 2A$.
5. Prove that

$$\tan (45^\circ + A) = \frac{1 + \sin 2A}{\cos 2A}, \quad \tan (45^\circ - A) = \frac{1 - \sin 2A}{\cos 2A}$$

6. If $A + B + C = 180^\circ$, prove that

$$\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C = 1$$

7. Find $\tan 67^\circ\ 30'$, and $\sin 22^\circ\ 30'$.
8. Prove that

$$\begin{aligned} \sin (A + B + C) = {} & \sin A (\cos B \cos C - \sin B \sin C) \\ & + \cos A (\sin B \cos C + \cos B \sin C) \end{aligned}$$

9. Prove that $2 \cos (45^\circ + \frac{1}{2} A) (\cos 45^\circ - \frac{1}{2} A) = \cos A$.
10. Prove that $\tan (45^\circ + \frac{1}{2} A) + \tan (45^\circ - \frac{1}{2} A) = 2 \sec A$.
11. Prove that $\tan \frac{1}{2} A + \cot \frac{1}{2} A = 2 \operatorname{cosec} A$.
12. Prove that $\cos^4 A - \sin^4 A = \sin 2A$.
13. Given $25 \sin A (\sin A - \cos A) = 4$, to find $\sin A$.
14. Let the notation $\tan^{-1} t$, stand for "the arc whose tangent is t;" prove that $\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = 45^\circ$ (See note at the end.)

[The notation used in this problem was first proposed by Sir John Herschel, and is now pretty generally adopted; thus $\sin^{-1} p$ means "the arc whose sine is p," $\cos^{-1} p$ means "the arc whose cosine is p;" and so on: if $\sin B = p$, then $\sin^{-1} p = B$, and if $\cos^{-1} q = A$, then $\cos A = q$, &c.]

PART III.

ON CERTAIN GENERAL THEOREMS AND DEVELOPMENTS IN ANALYTICAL TRIGONOMETRY.

124. We have divided the present compendious treatise into Three PARTS, because the subject itself embraces three divisions marked by distinct features, directed to distinct objects, and calling into operation distinct departments of algebraical investigation. The purport of the first part was to establish practical rules for the solution of triangles; that of the second part, to extend trigonometry to the theory of angles, unrestricted by a connection with a triangle; and the aim of this third part is to furnish a few theorems and developments of practical use only in a higher class of analytical inquiries, but possessing an amount of abstract interest and symbolical elegance sufficient to engage the attention and gratify the taste and curiosity of the mere theorist.

Developments of $\sin\theta$, $\cos\theta$ in series.

125. In this and subsequent investigations, the symbol θ will be employed to denote an angle, whether measured by degrees, or by the length of the trigonometrical arc that subtends it; that is, whether the measure be *gradual* or *arcual*. (See page 162.) You will at once perceive, in any particular case, which of these two kinds of measure is to be understood, or whether either, indifferently, may be taken; for as the two members of an equation must always be consistent with one another, if one side necessarily implies linear measure, so, of course, must the other side.

The object of the present article is to develope $\sin\theta$ and $\cos\theta$ in a series, proceeding according to the powers of the arc or angle θ itself; so that in each development, the numerical value of θ, in linear measure, must be taken, because the measure of $\sin\theta$ and $\cos\theta$ is so taken.

1. The series for $\sin\theta$ can involve only *odd* powers of θ. For since $\sin -\theta = -\sin\theta$, when $-\theta$ is put for θ, in the series, *all* the signs of the terms must be *changed*. Hence the series for $\sin\theta$ is of the form

$$\sin\theta = A_1\theta + A_3\theta^3 + A_5\theta^5 + \dots$$

which becomes 0 for $\theta = 0$, as it ought to do.

$$\therefore \frac{\sin \theta}{\theta} = A_1 + A_3\theta^2 + \dots\dots$$

And since when $\theta = 0$, we have $1 = A_1$

$$\therefore \sin\theta = \theta + A_3\theta^3 + A_5\theta^5 + \dots (1)$$

The condition $\frac{\sin 0}{0} = 1$, has been virtually assumed before (p. 162); if there be any doubt as to its truth, it may be removed by a reference to what is done at page 165, where it is shown that the smaller the arc, the more nearly does it approach to equality with its chord, the difference between the two continually approaching towards evanescence as the arc diminishes, and the ratio $\frac{\text{chord}\,\theta}{\theta}$ becoming unity when both actually vanish; so that, halving both arc and chord, we have $\frac{\sin 0}{0}=1$.

2. Since $\cos -\theta=\cos\theta$, the substitution of $-\theta$ for θ, can make no difference in the developed value of $\cos\theta$, which, moreover, must become 1, for $\theta=0$; hence the series for $\cos\theta$ must be of the form

$$\cos\theta=1+A_2\theta^2+A_4\theta^4+A_6\theta^6+\ldots.(2)$$

These preliminaries settled, put now $\theta=x+y$

$$\therefore\ \left.\begin{aligned}\sin\theta&=\sin x\cos y+\cos x\sin y\\ \cos\theta&=\cos x\cos y-\sin x\sin y\end{aligned}\right\}\ldots.(3)$$

But (1), (2),

$$\begin{aligned}\sin\theta&=(x+y)+A_3(x+y)^3+A_5(x+y)^5+\ldots.(4)\\ \cos\theta&=\quad 1\quad +A_2(x+y)^2+A_4(x+y)^4+\ldots.(5)\end{aligned}$$

Also, substituting (1), (2) in (3),

$$\left.\begin{aligned}\sin\theta=&(x+A_3x^3+A_5x^5+\ldots.)(1+A_2y^2+A_4y^4+\ldots.)\\ &+(y+A_3y^3+A_5y^5+\ldots.)(1+A_2x^2+A_4x^4+\ldots.)\end{aligned}\right\}\ldots.(6)$$

$$\left.\begin{aligned}\cos\theta=&(1+A_2x^2+A_4x^4+\ldots.)(1+A_2y^2+A_4y^4+\ldots.)\\ &-(x+A_3x^3+A_5x^5+\ldots.)(y+A_3y^3+A_5y^5+\ldots.)\end{aligned}\right\}\ldots.(7)$$

Now by the binomial theorem, the coefficient of y in (4) is

$$1+3A_3x^2+5A_5x^4+7A_7x^6+\ldots.$$

and the coefficient of y in (6) is

$$1+A_2x^2+A_4x^4+A_6x^6+\ldots.$$

and y being any value whatever, these series are therefore *equal*, whatever be the value of x (page 81).

In like manner, from (5) and (7), we have the equal series

$$\begin{aligned}&2A_2x+4A_4x^3+6A_6x^5+\ldots.\\ =&-x-\ A_3x^3-\ A_5x^5-\ldots.\end{aligned}$$

Hence, equating the coefficients of the like powers of x, we have the following conditions, namely,

$$\begin{aligned}2A_2&=-1 &\therefore\ A_2&=-\frac{1}{2}\\ 3A_3&=\ A_2 &\therefore\ A_3&=-\frac{1}{2\cdot3}\\ 4A_4&=-A_3 &\therefore\ A_4&=\frac{1}{2\cdot3\cdot4}\\ 5A_5&=\ A_4 &\therefore\ A_5&=\frac{1}{2\cdot3\cdot4\cdot5}\end{aligned}$$

and so on. Consequently

$$\left.\begin{aligned}\sin\theta&=\theta-\frac{\theta^3}{2\cdot3}+\frac{\theta^5}{2\cdot3\cdot4\cdot5}-\ldots.\\ \cos\theta&=1-\frac{\theta^2}{2}+\frac{\theta^4}{2\cdot3\cdot4\cdot5}-\ldots.\end{aligned}\right\}\ldots.(A)$$

We shall add here another series for $\sin\theta$, which is not without interest.

$$\begin{aligned}\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= 2^2\sin\frac{\theta}{2^2}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\\ &= 2^3\sin\frac{\theta}{2^3}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3},\ \text{and so on.}\end{aligned}$$

$$\therefore\ \sin\theta = 2^n\sin\frac{\theta}{2^n}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots.\cos\frac{\theta}{2^n}$$

Now, when $n=\infty$,

$$\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}=1\quad\therefore\quad\frac{\sin\frac{\theta}{2^n}}{\frac{1}{2^n}}=\theta=2^n\sin\frac{\theta}{2^n}$$

Hence the series for $\sin\theta$ is

$$\sin\theta=\theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots\ldots 1$$

For an important application of this result, see page 189.

The Theorem of Euler.

126. By the Exponential Theorem,

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{2\cdot3}+\frac{x^4}{2\cdot3\cdot4}+\ldots.$$

$$e^{-x}=1-x+\frac{x^2}{2}-\frac{x^3}{2\cdot3}+\frac{x^4}{2\cdot3\cdot4}-\ldots.$$

By addition and subtraction, dividing the result of each operation by 2, we have

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2}+\frac{x^4}{2\cdot3\cdot4}+\ldots\ldots\quad(1)$$

$$\frac{e^x-e^{-x}}{2}=x+\frac{x^3}{2\cdot3}+\frac{x^5}{2\cdot3\cdot4\cdot5}+\ldots.(2)$$

As these are all identical equations, we may put for the arbitrary symbol x anything we please; put $\theta\sqrt{-1}$ for x; then we have, observing that $\sqrt{-1}$ must enter every term of the series (2),

$$\frac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{2}=1-\frac{\theta^2}{2}+\frac{\theta^4}{2\cdot3\cdot4}-\ldots\ldots$$

$$\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}=\theta-\frac{\theta^3}{2\cdot3}+\frac{\theta^5}{2\cdot3\cdot4\cdot5}-\ldots.$$

But these series are respectively the developments of $\cos\theta$ and $\sin\theta$, as shown in the preceding article; hence we deduce these remarkable expressions of Euler, namely,

$$\left.\begin{aligned}\sin\theta&=\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}\\ \cos\theta&=\frac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{2}\end{aligned}\right\}\ldots\ldots(B)$$

by which we are to understand that if the imaginary exponentials are treated according to the same algebraical laws which govern the treatment of real quantities, the *algebraical* expressions on the right may always replace the *trigonometrical* expressions on the left.

The Theorem of De Moivre.

127. By Euler's Theorem just established, we have

$$\tfrac{1}{2}\left(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}\right)=\cos\theta$$

$$\tfrac{1}{2}\left(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}\right)=\sin\theta\cdot\sqrt{-1}$$

therefore, by addition and subtraction,

$$e^{\pm\theta\sqrt{-1}}=\cos\theta\pm\sin\theta\cdot\sqrt{-1}\ldots\ldots(1)$$

where θ is arbitrary. Put $n\theta$ for θ, n being anything,

$$\therefore\ e^{\pm n\theta\sqrt{-1}}=\cos n\theta\pm\sin n\theta\cdot\sqrt{-1}\ldots\ldots(2)$$

But $e^{\pm n\theta\sqrt{-1}}=(e^{\pm\theta\sqrt{-1}})^n=(\cos\theta\pm\sin\theta\cdot\sqrt{-1})^n$, by (1)

Hence by (2),

$$(\cos\theta\pm\sin\theta\cdot\sqrt{-1})^n=\cos n\theta\pm\sin n\theta\cdot\sqrt{-1}\ldots\ldots(C)$$

where n is quite unrestricted. And this is *the Theorem of De Moivre.*

We shall now exhibit an application or two of the preceding theorems.

Development of θ in a series according to the powers of tan θ.

128. By Euler's theorem,

$$e^{\theta\sqrt{-1}}=\cos\theta+\sin\theta\cdot\sqrt{-1}$$

$$e^{-\theta\sqrt{-1}}=\cos\theta-\sin\theta\cdot\sqrt{-1}$$

Dividing the first of these by the second,

$$e^{2\theta\sqrt{-1}}=\frac{\cos\theta+\sin\theta\cdot\sqrt{-1}}{\cos\theta-\sin\theta\cdot\sqrt{-1}}$$

$$=\frac{1+\tan\theta\cdot\sqrt{-1}}{1-\tan\theta\cdot\sqrt{-1}}$$

and taking the Napierian logarithm of each side, and dividing by 2,

$$\theta\sqrt{-1}=\tfrac{1}{2}\log\frac{1+\tan\theta\cdot\sqrt{-1}}{1-\tan\theta\cdot\sqrt{-1}}$$

$$=\tan\theta\ \sqrt{-1}+\frac{1}{3}(\tan\theta\cdot\sqrt{-1})^3+\frac{1}{5}(\tan\theta\cdot\sqrt{-1})^5+\ldots\ldots(\text{Alg. p. }97)$$

therefore, dividing by $\sqrt{-1}$,

$$\theta = \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots\ldots (1)^*$$

Let θ be the arcual measure of 45°; then $\tan\theta = 1$

$$\therefore \text{ arc } 45^\circ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots\ldots$$

The same series, for the length of the trigonometrical arc of 45°, may be got from Euler's theorem, in another way, thus—

By the theory of logarithms, page 102,

$$\log p = p - p^{-1} - \frac{p^2 - p^{-2}}{2} + \frac{p^3 - p^{-3}}{3} - \frac{p^4 - p^{-4}}{4} + \ldots\ldots$$

therefore, putting $p = e^{\theta\sqrt{-1}}$, we have, by Euler's theorem,

$$\theta\sqrt{-1} = 2\sqrt{-1}\cdot\sin\theta - \frac{2\sqrt{-1}\cdot\sin 2\theta}{2} + \frac{2\sqrt{-1}\cdot\sin 3\theta}{3} - \ldots\ldots$$

and dividing by $2\sqrt{-1}$,

$$\frac{\theta}{2} = \sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \ldots\ldots (2)\dagger$$

Let $\theta = 90°$; then we have, as above,

$$\text{arc } 45^\circ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots$$

and consequently the numerical value of the entire semicircumference π, radius being 1, is

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\ldots\right)$$

This series, however, is but ill fitted for actual summation, on account of its very slow convergency; but, from the general form (1), Euler has deduced other series which may be summed with comparatively little labour; thus

$$\text{since} \qquad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

if A and B together make up 45°, then $\tan(A + B) = 1$

$$\therefore \tan A + \tan B = 1 - \tan A \tan B$$

---

* In the development (1), the arc θ must not exceed 90°. If θ exceed this limit, the excess above 90°, being put for θ in the series, will obviously give the same values of the several terms (but—unless the arc terminate in an odd quadrant—with changed signs) as θ itself. A fuller examination of the peculiarities of such series as those marked (1) and (2) above, will be entered into in the *Differential Calculus*, in the third volume of this course.

† In the development (2) the arc θ must not exceed 180°. If θ exceed this limit, the excess above 180°, being put for θ in the series, will obviously give the same values for the several terms as θ itself, but the signs of the terms will be changed, unless the arc terminate in the first semicircle.

from which equation we may notice, in passing, that the tangents are such that their sum and product always make up *unity*.

Put $\tan A = \frac{1}{n}$ $\therefore$ $\frac{1}{n} + \tan B = 1 - \frac{\tan B}{n}$ $\therefore$ $\tan B = \frac{n-1}{n+1}$

Assume now $n = 2$; then

$$\tan A = \frac{1}{2},\ \tan B = \frac{1}{3}$$

Hence, since $A + B = 45°$, we have by (1)

$$\text{arc } 45° = \left\{ \begin{array}{l} \frac{1}{2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \dots. \\ + \frac{1}{3} - \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} - \frac{1}{7 \cdot 3^7} + \dots. \end{array} \right.$$

and both the series of which this development consists are rapidly convergent; but a series more convergent still was obtained by Machin; thus,

since $$\tan 2A - \frac{2 \tan A}{1 - \tan^2 A}$$

if $\tan A = \frac{1}{5}$, $\tan 2A = \frac{5}{12}$; therefore, since $\tan 45° = 1$, $2A$ must be less than 45°.

Again, from this value of $\tan 2A$, and the above formula, we have

$$\tan 4A = \frac{2 \tan 2A}{1 - \tan^2 2A} = \frac{120}{119}$$

so that $4A$ is *greater* than 45°.

Now (page 171) since $\tan 45° = 1$,

$$\tan(4A - 45°) = \frac{\tan 4A - 1}{1 + \tan 4A} = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \frac{1}{239}$$

Hence, putting $\frac{1}{5}$ for $\tan\theta$ in (1), we shall have the development of A; and putting $\frac{1}{239}$ for $\tan\theta$, the development of $4A - 45°$; so that, subtracting this latter development from 4 times the former, we shall get the development of 45°; that is,

$$\text{arc } 45° = \left\{ \begin{array}{l} 4\left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \dots\right) \\ -\left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \dots\right) \end{array} \right.$$

two series which are so convergent that if we take only eight terms in the first, and three in the second, we shall have, for the length of the semicircumference, the value

$$\pi = 3 \cdot 14159265358979$$

which is strictly correct as far as the decimals extend; and three terms of the first series, with one only of the second, will suffice to give the approximation commonly employed, namely,

$$\pi=3\cdot1416$$

In actually computing Machin's pair of series, as given above, it will be advisable to change the first into a form readily convertible into decimals, that is, to write it thus:

$$\frac{2}{10}-\frac{8}{3\cdot10^3}+\frac{32}{5\cdot10^5}-\frac{128}{7\cdot10^7}+\ldots\ldots$$

from which we see that any term being computed, we find the next by putting two 0's before it, multiplying the decimal thus formed by 4, and then dividing by whatever odd number 3, 5, 7, &c., forms the factor in the denominator of the term to be found.

129. The second of the two series may also be conveniently replaced by *two* others, obtained by splitting $\tan^{-1}\frac{1}{239}$ into two suitable arcs, or by making it the difference of two suitable arcs; thus, if $\frac{1}{70}$ be put for $\tan\alpha$, and $\frac{1}{99}$ for $\tan\beta$, we shall have

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

$$=\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\cdot\frac{1}{99}}=\frac{29}{1+70\cdot99}=\frac{29}{6931}$$

$$=\frac{1}{239}\quad\therefore\ \tan^{-1}\frac{1}{239}=\tan^{-1}\frac{1}{70}-\tan^{-1}\frac{1}{99}$$

Consequently the second of Machin's series may be replaced by

$$\frac{1}{70}-\frac{1}{3\cdot70^3}+\frac{1}{5\cdot70^5}-\ldots\ldots$$
$$-\left(\frac{1}{99}-\frac{1}{3\cdot99^3}+\frac{1}{5\cdot99^5}-\ldots\ldots\right)$$

that is, we may write

$$\text{arc } 45^\circ=\begin{cases}8\left(\frac{1}{10}-\frac{4}{3\cdot10^3}+\frac{16}{5\cdot10^5}-\ldots\ldots\right)\\-\left(\frac{1}{70}-\frac{1}{3\cdot70^3}+\frac{1}{5\cdot70^5}-\ldots\ldots\right)\\+\ \frac{1}{99}-\frac{1}{3\cdot99^3}+\frac{1}{5\cdot99^5}-\ldots\ldots\end{cases}$$

the last series in which may be rapidly computed, since $\frac{1}{99}$ is the circulating decimal $\cdot010101\ldots\ldots$, and the division by 9 and 11 is easy.*

---

* If seven terms of the first series, three of the second, and two of the third be taken, the length of the arc of 45° will be correctly ob-

But the actual calculation of the circumference of the circle, to any great extent of decimals, is *now* a matter of but little interest, though it is well to show how the work may be executed with but a very small portion of that vast labour by which the early investigators sought to accomplish the solution of the celebrated problem of the rectification of the circular circumference.*

Determination of the different roots of unity.

130. In De Moivre's theorem (C) page 182, put $\frac{1}{n}$ for n, and we have

$$(\cos\theta \pm \sin\theta \cdot \sqrt{-1})^{\frac{1}{n}} = \cos\frac{\theta}{n} \pm \sin\frac{\theta}{n} \cdot \sqrt{-1}$$

The first member of this equation is evidently unaltered, though

$2r\pi + \theta$ be written for θ,

since the sine and cosine of an arc remains undisturbed by however many entire circumferences that arc be increased. Making this substitution in the second member, we have

$$(\cos\theta \pm \sin\theta\sqrt{-1})^{\frac{1}{n}} = \cos\frac{2r\pi+\theta}{n} \pm \sin\frac{2r\pi+\theta}{n} \cdot \sqrt{-1} \ldots . (1)$$

Now, although the first member would have been unaffected by the proposed change, yet the second member admits, in consequence, of a variety of values, according to the number of units in n; we may conclude, therefore, that the nth root of $\cos\theta \pm \sin\theta \cdot \sqrt{-1}$ has *multiple values*, like as the square root of a quantity has twofold values. Let us illustrate this:—

Put $\theta = 0$; then the first number becomes simply $1^{\frac{1}{n}}$; and if n be now made $= 2, 3, 4$, &c., we shall have

$$\sqrt{1} = \cos\frac{2r\pi}{2} \pm \sin\frac{2r\pi}{2} \cdot \sqrt{-1} = \pm 1 : r = 0, 1$$

$$\sqrt[3]{1} = \cos\frac{2r\pi}{3} \pm \sin\frac{2r\pi}{3} \cdot \sqrt{-1} = 1,\ \frac{1}{2} \pm \frac{1}{2}\sqrt{-3} : r = 0, 1$$

$$\sqrt[4]{1} = \cos\frac{2r\pi}{4} \pm \sin\frac{2r\pi}{4} \cdot \sqrt{-1} = \pm 1,\ \pm\sqrt{-1} : r = 0, 1$$

$$\sqrt[5]{1} = \cos\frac{2r\pi}{5} \pm \sin\frac{2r\pi}{5} \cdot \sqrt{-1} = 1,\ \frac{\sqrt{5}-1}{4} \pm \frac{\sqrt{(-10-2\sqrt{5})}}{4},$$

$$-\frac{\sqrt{5}+1}{4} \pm \frac{\sqrt{(-10+2\sqrt{5})}}{4} : r = 0, 1, 2$$

&c. &c.

---

tained as far as ten places of decimals, an extent amply sufficient for every practical purpose.

* It is worthy of record, as a proof of the superior facilities of the modern methods, that, by aid of the series just given, the circumference of a circle, to radius 1, has been recently calculated to 530 places of decimals! This unprecedented amount of decimals has bee obtained *by Mr. William* Shanks, of Houghton-le-Spring, Durham.

We thus see that there are *three* different values for $\sqrt[3]{1}$, *four* for $\sqrt[4]{1}$, *five* for $\sqrt[5]{1}$, and so on. If other values for r, besides those above, had been taken, the results previously obtained would have reappeared; so that no additional values of the proposed roots of unity can exist. That the nth root of 1 has n values, and no more, may be thus shown:—

Let $r=0,\ 1,\ 2,\ 3\ldots,\ n-1$, in succession; then, when $\theta=0$, the arcs in the second member of (1) are the n following, namely,

$$0,\ \frac{1}{n}2\pi,\ \frac{2}{n}2\pi,\ \frac{3}{n}2\pi,\ldots\ldots\frac{n-1}{n}2\pi\ldots\ldots(2)$$

the fractions $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}$, &c., denoting what part of the whole circumference 2π each arc amounts to; and it is plain that, whatever whole number n may be, these fractions are all different; hence, since the n arcs are all different, and that each is less than a whole circumference, it follows that no two can have the same sine and cosine; there are thus n different values for $\sqrt[n]{1}$. That there cannot be a greater number of values, will appear by extending the series of arcs (2); for the sines and cosines of

$$\frac{n}{n}2\pi,\ \frac{n+1}{n}2\pi,\ \frac{n+2}{n}2\pi,\ \&c.\ \ldots\ldots(3)$$

are the same as the sines and cosines of (2), because 2π may be deducted from each of the latter without disturbing their terminations. It is thus evident that the nth root of unity consists of n different values, and that more than n roots cannot exist. As an nth root of any number is always that which, raised to the nth power, reproduces the number, and since, as just shown, if $N^{\frac{1}{n}}$ be the positive or ordinary arithmetical nth root of any number N, that $(\sqrt[n]{1}\cdot N^{\frac{1}{n}})^n$ is N, it follows that every number N has n different roots expressed by

$$\sqrt[n]{N}=\sqrt[n]{1}\cdot N^{\frac{1}{n}}$$

131. If instead of putting $\theta=0$, in the foregoing investigation, we had put $\theta=\pi$, the first member of (1) would have been -1; and, by imitating what is done above, we might get expressions for the second, third, fourth, &c. roots of -1. We here give, from De Morgan's "Trigonometry and Double Algebra," the twelve roots of $\sqrt[12]{1}$, and the twelve roots of $\sqrt[12]{-1}$; observing that, for the first set, we start with $\theta=0$; and, for the second set, with $\theta=\pi$.

	roots of $\sqrt[12]{1}$
$\cos 0^\circ \pm \sin 0^\circ\cdot\sqrt{-1}\ldots\ldots$	1
$\cos 30^\circ \pm \sin 30^\circ\cdot\sqrt{-1}\ldots\ldots$	$\frac{1}{2}\sqrt{3}\pm\frac{1}{2}\sqrt{-1}$
$\cos 60^\circ \pm \sin 60^\circ\cdot\sqrt{-1}\ldots\ldots$	$\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$
$\cos 90^\circ \pm \sin 90^\circ\cdot\sqrt{-1}\ldots\ldots$	$\pm\sqrt{-1}$
$\cos 120^\circ \pm \sin 120^\circ\cdot\sqrt{-1}\ldots\ldots$	$-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$
$\cos 150^\circ \pm \sin 150^\circ\cdot\sqrt{-1}\ldots\ldots$	$-\frac{1}{2}\sqrt{3}\pm\frac{1}{2}\sqrt{-1}$
$\cos 180^\circ \pm \sin 180^\circ\cdot\sqrt{-1}\ldots\ldots$	-1

	roots of $\sqrt[12]{-1}$
$\cos 15^\circ \pm \sin 15^\circ \cdot \sqrt{-1} \ldots\ldots$	$\frac{1}{4}(\sqrt{6}+\sqrt{2}) \pm \frac{1}{4}(\sqrt{6}-\sqrt{2})\sqrt{-1}$
$\cos 45^\circ \pm \sin 45^\circ \cdot \sqrt{-1} \ldots\ldots$	$\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{-2}$
$\cos 75^\circ \pm \sin 75^\circ \cdot \sqrt{-1} \ldots\ldots$	$\frac{1}{4}(\sqrt{6}-\sqrt{2}) \pm \frac{1}{4}(\sqrt{6}+\sqrt{2})\sqrt{-1}$
$\cos 105^\circ \pm \sin 105^\circ \cdot \sqrt{-1} \ldots\ldots$	$-\frac{1}{4}(\sqrt{6}-\sqrt{2}) \pm \frac{1}{4}(\sqrt{6}+\sqrt{2})\sqrt{-1}$
$\cos 135^\circ \pm \sin 135^\circ \cdot \sqrt{-1} \ldots\ldots$	$-\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{-2}$
$\cos 165^\circ \pm \sin 165^\circ \cdot \sqrt{-1} \ldots\ldots$	$-\frac{1}{4}(\sqrt{6}+\sqrt{2}) \pm \frac{1}{4}(\sqrt{6}-\sqrt{2})\sqrt{-1}$

In the above, the arcs, regularly increase by $\frac{1}{12}$ of the circumference, that is, by 30°; and, as soon as twelve results are obtained on the right, the series of arcs terminates, because, as shown above, the continuation of the series would only supply the roots over again.

Application of the developments (A) *to the construction of a table of sines and cosines.*

132. Let $\theta = \frac{90^\circ}{m}$; the linear measure of this arc will be $\frac{\pi}{2m}$, and the series (A), page 180, will be

$$\sin\frac{90^\circ}{m} = \frac{\pi}{2m} - \frac{1}{2\cdot 3}\left(\frac{\pi}{2m}\right)^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}\left(\frac{\pi}{2m}\right)^5 - \ldots\ldots$$

$$\cos\frac{90^\circ}{m} = 1 - \frac{1}{2}\left(\frac{\pi}{2m}\right)^2 + \frac{1}{2\cdot 3\cdot 4}\left(\frac{\pi}{2m}\right)^4 - \ldots\ldots$$

If $m = 90 \times 60$, we shall thus have the sine and cosine of $1'$; that is, the linear measure of the arc $1'$ being

$$\frac{\pi}{2m} = \frac{3\cdot 141592653589\ldots\ldots}{180 \times 60} = \cdot 0002908882\ldots\ldots$$

we shall have

$$\sin 1' = \cdot 0002908882 - \frac{\cdot 0002908882^{3*}}{2\cdot 3} + \ldots\ldots\ldots = \cdot 0002908882$$

$$\cos 1' = 1 - \frac{\cdot 0002908882^2}{2} + \frac{0002908882^4}{2\cdot 3\cdot 4} - \ldots\ldots = \cdot 9999999577$$

And, by help of these values, we may compute the sines and cosines of every minute in the quadrant from the formula

$$\sin(A+1') = 2\sin A\cos 1' - \sin(A-1')$$

A being made equal to $1'$, $2'$, $3'$, &c. in succession.

As $\cos 1'$ differs but very little from unity, it will be better to write, in this formula, $1-k$ for $\cos 1'$; k being its defect from unity; we shall thus have

$$\sin(A+1') = 2(\sin A - k\sin A) - \sin(A-1')$$

When the sines are computed up to $\sin 60^\circ$, the formula may be changed for the more convenient one at page 173, namely,

$$\sin(60^\circ + A) = \sin A + \sin(60^\circ - A)$$

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* In order to get the ten decimals on the right, the first term only of this series need be used; and in the second series, all the terms after the first fraction may be rejected.

and the table of sines and cosines completed. As all the other trigonometrical lines are derivable from these, we shall merely add that, when $\tan 45^\circ$ is reached, the calculation may be continued by means of the formula

$$\tan(45^\circ + A) = \tan(45^\circ - A) + 2\tan 2A \quad \text{(page 178, ex. 4.)}$$

The following is also useful for the secants, namely,

$$\sec A = \tfrac{1}{2}\{\tan(45^\circ + \tfrac{1}{2}A) + \tan(45^\circ - \tfrac{1}{2}A)\} \quad \text{(page 178, ex. 5.)}$$

$$= \tfrac{1}{2}\{\tan(45^\circ + \tfrac{1}{2}A) + \cot(45^\circ + \tfrac{1}{2}A\}$$

and for the cosecants, the formula

$$\operatorname{cosec} A = \tfrac{1}{2}(\tan \tfrac{1}{2}A + \cot \tfrac{1}{2}A) \quad \text{(page 178, ex. 11.)}$$

and many other facilitating expedients might be pointed out; but as tables sufficiently trustworthy already exist, there is no necessity for any recomputation; and what is here given is merely intended to show the leading principles upon which tables are constructed.

To prove that the decimals in the number π are interminable.

123. It was shown at page 181 that

$$\sin\theta = \theta \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\ldots.1$$

If we make $\theta = \frac{\pi}{2}$, we shall have $\sin\theta = 1$; and since $\text{secant} = \frac{1}{\text{cosine}}$, if each member of the preceding equation be divided by the cosines, we shall get for $\frac{\pi}{2}$, the following expression, namely

$$\frac{\pi}{2} = \sec\frac{\pi}{2}\sec\frac{\pi}{2^2}\sec\frac{\pi}{2^3}\sec\frac{\pi}{2^4}\ldots.1 \qquad \text{(A)}$$

As the arc diminishes, by the continual bisections of it, the secant differs from 1 by a decimal, the first significant figure of which becomes more and more remote till, at length, the significant decimals commence at a place infinitely distant.

The *n*th secant is not only less than every preceding one, but, as a necessary consequence, it is also less than the product of all the preceding secants; and the *n*th secant has an infinite number of decimals. Consequently the product of all must have an infinite number of decimals, and therefore the decimals in π must be interminable; so that the diameter and circumference of a circle are numerically incommensurable quantities.

Every one of the factors in (*A*) is irrational; for (page 174.)

$$\cos\tfrac{1}{2}A = \sqrt{\frac{1+\cos A}{2}}$$

and since

$$\cos\frac{\pi}{2} = \tfrac{1}{2}\sqrt{2} \quad \text{(page 177.)}$$

$$\therefore \cos\frac{\pi}{2^2} = \sqrt{\frac{1+\frac{1}{2}\sqrt{2}}{2}} = \tfrac{1}{2}\sqrt{(2+\sqrt{2})}$$

$$\cos\frac{\pi}{2^3}=\sqrt{\frac{1+\frac{1}{2}\sqrt{(2+\sqrt{2})}}{2}}=\tfrac{1}{2}\sqrt{[2+\sqrt{(2+\sqrt{2})}]}$$

$$\cos\frac{\pi}{2^4}=\sqrt{\frac{1+\frac{1}{2}\sqrt{[2+\sqrt{(2+\sqrt{2})}]}}{2}}=\tfrac{1}{2}\sqrt{\{2+\sqrt{[2+\sqrt{(2+\sqrt{2})}]}\}}$$

&c. &c.

$$\therefore\ \sec\frac{\pi}{2}=\frac{2}{\sqrt{2}}$$

$$\sec\frac{\pi}{2^2}=\frac{2}{\sqrt{(2+\sqrt{2})}}$$

$$\sec\frac{\pi}{2^3}=\frac{2}{\sqrt{[2+\sqrt{(2+\sqrt{2})}]}}$$

$$\sec\frac{\pi}{2^4}=\frac{2}{\sqrt{\{2+\sqrt{[2+\sqrt{(2+\sqrt{2})}]}\}}}$$

and so on continually; the numerator of every factor being 2, and the denominator of each being derived from the denominator of the preceding by prefixing 2, and then putting a radical sign $\surd$, before the whole; so that the denominator of the factor $\sec\frac{\pi}{2^n}$ will involve n radical signs.

The mere fact that an additional term, and an additional extraction, is introduced into every succeeding denominator, precludes the possibility of the radicals being neutralized by the multiplication of the denominators together.

In terminating this volume, it may be well to notice that, in most of the general developments of analysis, *unit-factors* are suppressed; and that, on this account, it frequently happens—when fractional exponents enter—that there is an apparent discrepancy between the developed and the undeveloped forms; the latter appearing to have a multiplicity of values, and the former but a single value. The defect may always be supplied by introducing, as a factor, the suitable roots of unity, as already noticed in the Algebra, page 102, article 65. Thus, in the exponential theorem, if the exponent be a fraction, the usual form

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2\cdot 3}+\ldots\ldots$$

requires the correction adverted to; and to render the two members of the identity perfectly consistent with each other, whether x be integral or fractional, we should write the form thus

$$e^x=(1+x+\frac{x^2}{2}+\frac{x^3}{2\cdot 3}+\ldots.)1^x$$

and a like modification is necessary in all similar cases.

NOTE, *referred to at page* 178.

The notation in the text is also employed in connection with logarithmic as well as trigonometrical forms; thus, $\log^{-1}p$, means the number whose log is p. As an illustration of the convenience of this notation, we shall here give a curious theorem due to Euler.

From $$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

we have $$A-B=\tan^{-1}\frac{\tan A-\tan B}{1+\tan A\tan B}\dots(1)$$

or putting t_1 for $\tan A$, and t_2 for $\tan B$, we have

$$A=\tan^{-1}t_1,\text{ and }B=\tan^{-1}t_2$$

hence, by substitution in (1),

$$\tan^{-1}t_1-\tan^{-1}t_2=\tan^{-1}\frac{t_1-t_2}{1+t_1t_2}$$

Now the following identity is obvious, namely,

$$A_1=(A_1-A_2)+(A_2-A_3)+(A_3-A_4)+\dots+(A_{n-1}-A_n)+A_n$$

putting therefore $A_1=\tan^{-1}t_1$, $A_2=\tan^{-1}t_2$, $A_3=\tan^{-1}t_3$, &c., we have the following theorem, namely,

$$\tan^{-1}t_1=\tan^{-1}\frac{t_1-t_2}{1+t_1t_2}+\tan^{-1}\frac{t_2-t_3}{1+t_2t_3}+\tan^{-1}\frac{t_3-t_4}{1+t_3t_4}+\dots+\tan^{-1}t_n\dots(2)$$

A similar theorem results from the identity

$$A_1=(A_1+A_2)-(A_2+A_3)+(A_3+A_4)-(A_4+A_5)+\dots\pm(A_{n-1}+A_n)\pm A_n$$

namely, the theorem

$$\tan^{-1}t_1=\tan^{-1}\frac{t_1+t_2}{1-t_1t_2}-\tan^{-1}\frac{t_2+t_3}{1-t_2t_3}+\tan^{-1}\frac{t_3+t_4}{1-t_3t_4}-\dots\pm\tan^{-1}t_n\dots(3)$$

Let $t_1=1$, $t_2=\frac{1}{2}$, $t_3=\frac{1}{3}=t_n$, then (2) becomes

$$\begin{aligned}\text{arc}\,45^\circ&=\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{3}\\&=2\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{7}\end{aligned}$$

For $t_1=1$, the several fractions may be assumed of any values whatever, and thence t_2, t_3, &c., and finally t_n, may be determined. And, take whatever value we please for t_1, and then whatever value we please for each of the several fractions, the quantities t_2, t_3, &c., up to t_n, suited to those assumed values, may always be readily discovered.

ANSWERS

TO

THE EXERCISES IN THE ALGEBRA.

ADDITION. Page 6.

Ex. (1) $9ax+7b$. (2) $11acx+7m$. (3) $11xyz-2ab$.
(4) $5ny$. (5) $8x-3y+z$. (6) $7ax-10cz$.
(7) $-7mz-13nx+9$. (8) $3ax-by-2$.
(9) $6b+2c-\frac{3}{4}d-10\frac{1}{4}$. (10) $12ax-10by-4$.
(11) $-mz-3x-1$. (12) $6\frac{3}{8}axy-\frac{3}{4}bz$.

Page 8.

(1) $3ax-12by+80$. (2) $31-8ax+7mz+xy$.
(3) $7abx+75cy-16abc-87$. (4) $11xy-27yz+15xyz+5xz$.
(5) $-5az$. (6) $\frac{1}{4}ax+9\frac{3}{4}b+16\frac{1}{2}$.
(7) $2ax-16b-10$. (8) $-3xy+10az-16$.
(9) $17xy-9az+10mn-13$. (10) $px+15qy-7c+3$.

SUBTRACTION. Page 10.

(1) $11xy-3az-6b+3$. (2) $-18xyz+12ab-5cd+4$.
(3) $6mx+2ny+8pz-15+a$. (4) $9z+by+2a+c-3m$.
(5) $17az+by-5cx+6$. (6) $-18px+23qy-4r-11$.
(7) $16by+13x+az-20$. (8) $x-10c+6b-14a+9$.
(9) $-25az+2\frac{1}{2}xy+6$. (10) $-\frac{x}{a}+6\frac{x}{b}-5y-4$.
(11) $12ax-22by+1\frac{1}{8}c-27$. (12) $8\frac{3}{4}y-7\frac{1}{2}z-2\frac{1}{2}x+1\frac{5}{8}$.
(13) $2x+3ay+9z+\frac{1}{4}$. (14) $3\frac{3}{4}ay-5\frac{1}{2}cex+3\frac{1}{4}z-1$.
(15) $4x-2\frac{2}{3}y+5\frac{1}{2}z+a-b-c$. (16) $-\frac{1}{2}z-x+y+57\frac{1}{4}$.

SIMPLE EQUATIONS. Page 17.

(1) $x=5$. (2) $x=3$. (3) $x=24$. (4) $x=12$.
(5) $x=4$. (6) $x=1\frac{13}{47}$. (7) $x=7$. (8) $x=4$.

Ex. (9) $x=2\frac{3}{11}$. (10) $x=14\frac{4}{7}$. (11) $x=8$.
(12) $x=3$. (13) $x=-1\frac{13}{17}$. (14) $x=38$.
(15) $x=16\frac{2}{3}$. (16) $x=5\frac{2}{7}$. (17) $x=5\frac{1}{3}$.
(18) $x=-3\frac{1}{4}$. (19) $x=5\frac{4}{13}$. (20) $x=4\frac{1}{3}$.

Questions for Exercise. Page 21.

(1) 2 and 8. (2) 200. (3) 15 and 5.
(4) A, 66*l.* 13*s.* 4*d.*; B, 33*l.* 6*s.* 8*d.*; C, 200*l.*
(5) 100lb. (6) 18 days.
(7) 72lb. (8) 75 gallons.
(9) 90 half-guineas, 11 crowns.
(10) 5*l.* (11) 1 hour, $52\frac{1}{2}$ minutes.
(12) $\frac{30}{31}$ of a day. (13) 27 min. $16\frac{4}{11}$ sec. past V.
(14) 32 min. $48\frac{7}{11}$ sec. past XII.
(15) The number is 4. (16) $23\frac{1}{3}$ days.
(17) 24 miles. (18) 45 miles and 105.
(19) 2 and 8. (20) $2\frac{2}{7}$ days.

Page 24.

(1) $7\{(a+x)^2+1\}$. (2) $4\sqrt{(a-x)}+6$.
(3) $3(2ax+3)^{\frac{1}{3}}+5b$. (4) $\sqrt{(x^2-y^2)}-5x+9a$.
(5) $6\sqrt{(x^2-y^2)}-a\sqrt{(x^2+y^2)}$.

MULTIPLICATION. Page 25.

(1) $-12a^2b^3x^4y^4$. (2) $30ab^4x^4y^5z^3$. (3) $-30a^3x^6y$.
(4) $-120ab^3x^5y^2$. (5) $-\frac{3}{2}a^3x^3y^3$. (6) $-\frac{1}{6}a^2bx^2y^2z^2$.

Page 26.

(1) $-10ax^3y^2+15a^2x^2z^3$.
(2) $-84a^4b^5x^3+21ab^3cx^2y^3-14ab^3x^2$.
(3) $12a^3b^4x^4-4a^2b^2x^5y^2$. (4) $-21bx^3y^5-3abxy^3z^2+12bxy^3$
(5) $-63x^6y^4z^5+70x^3y^8z^5+84x^3y^4z^{10}$.
(6) $8a^3bx^4+8a^2bx^3$. (7) $28a^3bxy-16aby$.
(8) $a^2bx^2y^3-2a^3x^3+\frac{3}{2}a^2x^2y^2$. (9) $5a^2bcx^2+ab^3cxy-abc^2xz$.
(10) $-7a^2bx^{\frac{1}{2}}y^2z^3+ab^2y^3z^3$.

Page 27.

Ex.(1) $6a^2x^2-2ax-8$.

(2) $48ax^4-3(12a+4b)x^3+(9b+4c)x^2-3cx$.

(3) $16x^4y-6x^3y+4(6-8y)x^2+12xy-48$.

(4) $18ax^3+3(5a-2)x^2-(18a+5)x+6$.

(5) $-140x^3-79x^2+78x+45$. (6) $x^3-4xy^2+2x^2y-8y^3$.

(7) $a^3+3a^2x+3ax^2+x^3$. (8) $a^3-3a^2x+3ax^2-x^3$.

(9) $6x^2-9x$. (10) $8x^3-3x^2+1\frac{7}{12}x-\frac{1}{6}$.

(11) $\frac{1}{4}x^3-\frac{7}{12}x^2y+\frac{4}{3}xy^2-\frac{1}{3}y^3$. (12) $7x^3-39x^2+38x+24$.

(13) $15a^2x^2-26abxy-19ax+8b^2y^2+16by+6$.

(14) $42x^3-63\frac{1}{2}x^2+5\frac{1}{2}x+6$.

(15) $4a^4x^4+4a^3bx^3-3a^2b^2x^2-4ab^3x-b^4$.

(16) $14x^5-29x^4-8x^3+19x^2+3x-2$.

(17) $x^4-(a+b)^2x^2+2(a+b)cx+c^2$.

(18) The product is $x^4-y^4=(x^2+y^2)\,(x^2-y^2)$.

(19) The product is $a^6-x^6=(a^3+x^3)\,(a^3-x^3)$.

(20) $(x^2+y^2)\,(x^2-y^2)=x^4-y^4$, and $(x^4-y^4)\,(x^4+y^4)=x^8-y^8$.

DIVISION. Page 30.

(1) $\frac{14ax^2y^3}{-2axy^3}=-7x$. (2) $-\frac{a^4y^6z^2}{3a^3y^2x^3}=-\frac{ay^4z^2}{3x^3}$.

(3) $\frac{-12b^7x^4y}{-6a^3b^5x^4y^2}=\frac{2b^2}{a^3y}$. (4) $\frac{5ax^3y^5}{3a^2xyz^4}=\frac{5x^2y^4}{3az^4}$.

(5) $\frac{16a^4x^2y^{\frac{1}{2}}}{-4a^3x^3\sqrt{y}}=-\frac{4a}{x}$. (6) $-\frac{7xy^4\sqrt[3]{z}}{6x^2y^4z^{\frac{1}{3}}}=-\frac{7}{6x}$.

(7) $\frac{3a^2m^3xy}{5a^4n^2y^3}=\frac{3m^3x}{5a^2n^2y^2}$. (8) $\frac{2b^7x^4z^{\frac{1}{3}}}{-\frac{1}{2}x^5yz^{\frac{1}{3}}}=-4\frac{b^7}{xy}$.

(9) $\frac{-7x^{\frac{1}{2}}yz^{\frac{1}{3}}}{-2\sqrt[3]{z}}=\frac{7x^{\frac{1}{2}}y}{2}$. (10) $\frac{\frac{1}{4}abc^2\sqrt{x}}{\frac{1}{2}a^2b^2cx^{\frac{1}{2}}}=\frac{c}{2ab}$.

(11) $\frac{9a^2n^2\sqrt{(x+y)}}{18a^3n\sqrt{(x-y)}}=\frac{n\sqrt{(x+y)}}{2a\sqrt{(x-y)}}$.

(12) $\frac{3ab(x-y)^{\frac{1}{3}}}{6a^2b\sqrt[3]{(x-y)}}=\frac{1}{2a}$.

Page 30.

(1) $\frac{8a^2x^3-12ax^2}{2ax}=4ax^2-6x$.

(2) $\frac{12m^4x^6-8m^3x^5+16m^2x^4}{4m^2x^3}=3m^2x^3-2mx^2+4x$.

Ex.(3) $\dfrac{18c^5x^4y^8-3c^3x^3y^5-6c^2x^2y^3}{3c^2xy^2}=6c^3x^3y^6-cx^2y^3-2xy.$

(4) $\dfrac{5a^5b^6x^7+10a^4b^7x^6-2}{5a^4b^3x^2}=ab^3x^5+2b^4x^4-\dfrac{2}{5a^4b^3x^2}.$

(5) $\dfrac{3x^2y^3z^2-4x^2y^2z^3-xyz^4}{5x^2y^2z}=\dfrac{3yz-4z^2}{5}-\dfrac{z^3}{5xy}.$

(6) $\dfrac{12x^5y^7-3x^4-6x^3y^2}{3x^2y}=4x^3y^6-2xy-\dfrac{x^2}{y}.$

(7) $\dfrac{15-9a^3x^4-9b^2x^3y^3+27}{5x^2y^3}=\dfrac{3}{x^2y^3}-\dfrac{9a^3x^2}{5y^3}-\dfrac{9b^2x}{5}+\dfrac{27}{5x^2y^3}.$

(8) Cancelling $4x$,

$$\frac{a^4x^7-2a^3bx-2a^5bx+6bx^4}{ab}=-2a^2x-2a^4x+\frac{a^3x^7}{b}+\frac{6x^4}{a}.$$

(9) Cancelling ax, $\dfrac{5\,(3a^2x^5-2)\,(7a^4x+3)-10a^2x^2}{2ax}=$

$$\frac{105a^6x^6+45a^2x^5-70a^4x-30}{2ax}-5ax=52\tfrac{1}{2}a^5x^5+22\tfrac{1}{2}ax^4-35a^3-5ax-\frac{15}{ax}$$

(10) $\dfrac{(a^3-2a^2+2x^2)\,(4a^4-3x^2)}{2ax^2}=\dfrac{4a^7-8a^6+8a^4x^2-3a^3x^2+6a^2x^2-6x^4}{2ax^2}=$

$$4a^3+3a+\frac{2a^6-4a^5}{x^2}-\frac{3a^2}{2}-\frac{3x^2}{a}.$$

Page 33.

(1) $5x^3+4x^2+3x+2.$

(2) a^2-ax+x^2, and $a^2+ax+x^2+\dfrac{2x^3}{a-x}.$

(3) $x^3-3ax^2+3a^2x-a^3.$ (4) $24z^2-12az-50a^2.$

(5) $9x^4-10x^3+5x^2-30x.$ (6) $x^4-x^3+x^2-x+1-\dfrac{2x}{x^2+x-1}.$

(7) $2x-4+\dfrac{2x-2}{x^2-1}=2x-4+\dfrac{2}{x+1}.$

(8) $\dfrac{1}{3}x+2.$

(9) The remainder is 5; and when 2 is put for x in the dividend, we get $8-20+14+3=5$, like the remainder.

(10) The remainder is a^3+ba^2+ca+d, which is the same as the dividend when a is put for x.

Page 38.

(1) $\sqrt{a^2x^6y^8}=ax^3y^4.$ (2) $\sqrt{9a^4x^2y}=3a^2x\sqrt{y}.$

(3) $12a^2x^3y=2ax\sqrt{3xy}.$ (4) $\sqrt{\dfrac{a^2}{b^3}}=\dfrac{a}{b^2}\sqrt{b}.$

Ex. (5) $\sqrt[3]{a^4x^4y^6}=axy^2\sqrt[3]{ax}$. (6) $\sqrt{8a^3x^4y^2}=2ax\sqrt[3]{xy^2}$.

(7) $\sqrt[3]{27b^4c^5x^6}\sqrt[3]{\frac{a^4}{b^5}}=\frac{3acx^2}{b}\sqrt[3]{ab^2c^2}$.

(8) $\sqrt{\frac{2x^5}{3y^6}}=\frac{x^2}{3y^3}\sqrt{6x}$. (9) $(16a^4b^2y^6)^{-\frac{1}{2}}=\frac{1}{4a^2by^3}$.

(10) $a(x^2y^3)^{-\frac{1}{3}}=ay^{-1}x^{-\frac{2}{3}}$.

(11) $(288x^5y^7)^{-\frac{1}{2}}=\frac{x^{-2}y^{-3}(2xy)^{-\frac{1}{2}}}{12}$.

(12) $(-8ax^3y^5)^{-\frac{1}{3}}=-\frac{1}{2}(xy)^{-1}(ay^2)^{-\frac{1}{3}}$.

(13) $\sqrt{\frac{63a^4x^3}{50y^5}}=\frac{9a^2x}{10y^2}\sqrt{14xy}$.

(14) $\left(\frac{1}{3}\sqrt{3}\right)^3=\frac{1}{9}\sqrt{3}$. (15) $\left(\frac{1}{2}\sqrt{2}\right)^4=\frac{1}{4}$.

(16) $\sqrt{(16\sqrt{6})}=4\sqrt[4]{6}$. (17) $6^{\frac{3}{2}}=6\sqrt{6}$.

(18) $\left(\frac{1}{8}a^3b\right)^{\frac{1}{3}}=\frac{1}{2}a\sqrt[3]{b}$; $\sqrt[m]{x^{\frac{1}{n}}}=x^{\frac{1}{mn}}$

FRACTIONS. Page 39.

(1) $\frac{x-3a}{x}$. (2) $\frac{7abc-m-1}{2c}$.

(3) $\frac{23a+10ax-2x^3}{5a}$. (4) $\frac{4x^3-25x^2+20x+25}{4x+3}$.

(5) $\frac{2x^4}{x^2+a^2}$. (6) $\frac{a^3+x^3}{a-x}$.

(7) $\frac{a^2+b^2+c^2}{2ab}$. (8) $\frac{(a+b)^2}{ab}$; $\frac{(a-b)^2}{ab}$.

Page 39.

(1) $a+b+\frac{c}{a-b}$. (2) $x^3+y^3-\frac{c}{x^3-y^3}$.

(3) $2(x^2+xy+y^2)$. (4) $2a-1-\frac{7}{4a}$.

(5) $2(x^2-a^2)+\frac{2a^4}{x^2+a^2}$. (6) $5-\frac{5(a^2+2a+4)}{3a^3+2a^2-2a-4}$.

(7) $x-y$. (8) $x^2+3x+\frac{7x(x-2)}{x^2-3x+2}$.

(9) $2(a-2x)+\frac{10x^2}{a+2x}$.

Page 41.

Ex.(1) $\frac{4bx}{2ab}, \frac{3a^2+ax}{2ab}$. (2) $\frac{a^2-b^2}{2c(a-b)}, \frac{2bc}{2c(a-b)}$.

(3) $\frac{2ab(a-b)}{a^2-b^2}, \frac{(1-b)(a+b)}{a^2-b^2}$. (4) $\frac{a(x+y)}{x^2-y^2}, \frac{b}{x^2-y^2}$.

(5) $\frac{a(x-y)}{x^2-y^2}, \frac{b(x+y)}{x^2-y^2}, \frac{c}{x^2-y^2}$. (6) $\frac{7(3x^2-4a)}{35}, \frac{20}{35}$.

(7) $\frac{ax(1+x)}{(1-x)^2}, \frac{a}{(1-x)^2}$. (8) $\frac{a+x}{a^2-x^2}, \frac{(a-x)^2}{a^2-x^2}, \frac{a+x}{a^2-x^2}$.

(9) $\frac{3x^4-2x^3-5x^2+4x-2}{x^4-6x^2+8}, \frac{2x^3+x^2-8x-4}{x^4-6x^2+8}$.

(10) $\frac{bc(2x+3)}{abc}, \frac{ac(3x-2)}{abc}, \frac{3ab}{abc}$.

(11) $\frac{4(x-1)^2}{4(x^2-1)}, \frac{4(x+1)^2}{4(x^2-1)}, \frac{3(x^2-1)}{4(x^2-1)}, \frac{20(x^2-1)}{4(x^2-1)}$.

ADDITION AND SUBTRACTION OF FRACTIONS. Page 41.

(1) $\frac{47a}{60}$. (2) $\frac{(x+b)^2}{x(x^2+b^2)}$. (3) $\frac{2(x^2+y^2)}{x^2-y^2}$.

(4) $\frac{13x-5}{4}$. (5) $\frac{x+y-1}{x^2-y^2}$. (6) $\frac{20ax+15bx-36x}{60}$.

(7) $\frac{3x^2+5x+2}{x^2-1}$. (8) $\frac{4xy}{x^2-y^2}$. (9) $\frac{2b^2}{a^2+b^2}$.

(10) 0. (11) $\frac{a^4-2ax-x^4}{a^2-x^2}$.

MULTIPLICATION AND DIVISION OF FRACTIONS. Page 42.

(1) $\frac{3axy}{35}$. (2) $\frac{12(a^2-b^2)}{25}$. (3) $\frac{20}{ax}$.

(4) $\frac{6}{y}$. (5) $4(x^2+ax+a^2)$. (6) $\frac{x}{y}-\frac{a}{b}=\frac{bx-ay}{by}$.

(7) $\frac{a^2-x^2}{ax}$. (8) $\frac{a^2+x^2}{a^2-x^2}$. (9) $\frac{(a^2-x^2)^2}{ax(a^2+x^2)}$.

(10) 1. (11) $\frac{2(2a+b)^2}{9a^2-4b^2}$.

(12) $\frac{21}{20\sqrt{(a+x)}}=\frac{21\sqrt{(a+x)}}{20(a+x)}$.

SIMPLE EQUATIONS. Page 46.

Ex.(1) $x=36$. (2) $x=18$. (3) $x=60$.
(4) $x=12$. (5) $x=2a$. (6) $x=64$.
(7) $x=5\frac{3}{8}$. (8) $x=9$. (9) $x=6\frac{1}{2}$.
(10) $x=5$. (11) $x=\frac{a-b}{2\sqrt{ab}}=\frac{(a-b)\sqrt{ab}}{2ab}$.
(12) $x=a-1$. (13) $x=2$. (14) $x=4\frac{7}{8}$.
(15) $x=\frac{4am}{(m+1)^2}$. (16) $x=1\frac{1}{32}$. (17) $x=3$.
(18) $x=1$. (19) $x=\frac{4a}{5}$. (20) $x=3$.

Questions in Simple Equations. Page 47.

(3) 47. (4) 25. (5) 4*l.* 10*s.* (6) 7.
(7) 22, 31, 9 and 54. (8) $4\frac{7}{12}$ miles an hour.
(9) 71 gal. at 9*s.* 6*d.*, and 33 at 13*s.* 6*d.*
(10) $\frac{abc}{ab+ac+bc}$ minutes. (11) 49 and 42.
(12) $6\frac{2}{3}$ gal. of the first, and $13\frac{1}{3}$ of the second.

SIMPLE EQUATIONS, with Two Unknown Quantities. Page 51.

(1) $x=4$, $y=5$. (2) $x=7$, $y=5$. (3) $x=7$, $y=4$.
(4) $x=10$, $y=7$. (5) $x=4$, $y=3$. (6) $x=3$, $y=8$.
(7) $x=\frac{a^2+b}{2a}$, $y=\frac{a^2-b}{2a}$. (8) $x=\frac{1}{2}$, $y=\frac{1}{3}$.
(9) $x=30$, $y=19$. (10) $x=21\frac{3}{7}$, $y=50$.
(11) $x=21$, $y=63$. (12) $x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}$, $y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$.
(13) $x=\frac{bc}{a+b}$, $y=\frac{ac}{a+b}$. (14) $x=12$, $y=5$.

Questions. Page 52.

(1) A.'s share 22*l.*; B.'s share 26*l.*
(2) 8 and 16. (3) $\frac{4}{21}$.
(4) 1*l.* 12*s.*; 14 persons. (5) 23.
(6) 20 bush. rye, 52 bush. wheat.
(7) 12 persons; 3*l.* (8) 6 min. and 4 min.
(9) 21 (10) 22.

EQUATIONS, with Three Unknown Quantities. Page 55.

Ex. (1) $x=4$, $y=6$, $z=8$. (2) $x=3$, $y=2$, $z=1$.
(3) $x=10$, $y=2$, $z=3$. (4) $x=7$, $y=10$, $z=9$.
(5) A.'s share, 50*l.*, B.'s 30*l.*, C.'s 20*l.*
(6) Brandy, 15 gal.; wine, 24 gal.; cyder, 21 gal.
(7) 6, 12, 3, and 27. (8) 12, 60, and 80.
(9) A. $14\frac{34}{49}$ days; B. $17\frac{23}{41}$ days; C. $23\frac{7}{31}$ days.

Examples in QUADRATIC EQUATIONS. Page 60.

(1) $x=3$, or -1. (2) $x=5$, or 3. (3) $x=3$, or 14.
(4) $x=\frac{1}{2}(1\pm\sqrt{5})$. (5) $x=6$, or $5\frac{2}{3}$. (6) $x=1\pm\sqrt{-1}$.
(7) $x=\dfrac{3\pm\sqrt{7}}{2}$. (8) $x=4$, or 9. (9) $x=4$, or -21.
(10) $x=60$, or 235. (11) $x=1$, or 4. (12) $x=1$, or $(-\frac{1}{2})^{\frac{2}{3}}$.
(13) $x=-1$, or $-\frac{1}{2}$. (14) $x=a$.
(15) $x=5$, or $37\frac{1}{2}$. (16) $x=10$, or $-2\frac{1}{5}$.
(17) $x=8$, or $(-\frac{27}{8})^{\frac{2}{3}}$. (18) $x=9$, or $-\frac{25}{81}$.

Questions. Page 63.

(1) 6 and 27. (2) 40.
(3) 2 and 4, or -8 and 14. (4) Impossible.
(5) A. 6 miles an hour; B. 5 miles.
(6) 15. (7) 7.
(8) 8 and 15, or -10, and -12.
(9) 18*s.* (10) $2\pm\dfrac{4\sqrt{3}}{3}$, and $-2\pm\dfrac{4\sqrt{3}}{3}$.

SIMULTANEOUS QUADRATICS. Page 67.

(1) $\begin{cases} x=\pm 7, \text{ or } \pm\dfrac{11\sqrt{2}}{2}. \\ y=\pm 4, \text{ or } \pm\dfrac{3\sqrt{2}}{2}. \end{cases}$ (2) $\begin{cases} x=\pm 2, \text{ or } \pm\dfrac{3\sqrt{7}}{7}. \\ y=\pm 1, \text{ or } \mp\dfrac{8\sqrt{7}}{7}. \end{cases}$

(3) $\begin{cases} x=\pm 3, \text{ or } \pm\dfrac{4\sqrt{6}}{3}. \\ y=\pm 1, \text{ or } \pm\dfrac{\sqrt{6}}{6}. \end{cases}$ (4) $\begin{cases} x=\pm 6, \text{ or } \pm 102. \\ y=\pm 5, \text{ or } \mp 59. \end{cases}$

(5) $\begin{cases} x=\pm 2, \text{ or } \pm\dfrac{5}{31}\sqrt{31}. \\ y=\pm 3, \text{ or } \mp\dfrac{6}{31}\sqrt{31}. \end{cases}$ (6) $\begin{cases} x=\pm 3, \text{ or } \pm\dfrac{\sqrt{2}}{6}. \\ y=\pm 2, \text{ or } \mp\dfrac{7\sqrt{2}}{6}. \end{cases}$

SYMMETRICAL QUADRATICS. Page 68.

Ex. (1) $\begin{cases} x=4, \text{ or } 8. \\ y=8, \text{ or } 4. \end{cases}$ (2) $\begin{cases} x=1\pm\frac{\sqrt{3}}{3}. \\ y=1\mp\frac{\sqrt{3}}{3}. \end{cases}$ (3) $\begin{cases} x=4, \text{ or } 5. \\ y=5, \text{ or } 4. \end{cases}$

(4) $\begin{cases} x=2, \text{ or } 3, \text{ or } -3\pm\sqrt{3}. \\ y=3, \text{ or } 2, \text{ or } -3\mp\sqrt{3}. \end{cases}$

ARITHMETICAL PROGRESSION. Page 73.

(1) 190.
(2) 0.
(3) 3, 5, 7, 9, &c.
(4) 5 and 7.
(5) 11, 16, and 21.
(6) *Ten.*
(7) 280 or 140.
(8) 11 miles 840 yards.

GEOMETRICAL PROGRESSION. Page 76.

(1) 4095.
(2) 265720.
(3) 4, 8, 16.
(4) $\pm1, \frac{1}{2}, \pm\frac{1}{4}, \frac{1}{8}, \pm\frac{1}{16}$.
(5) $\Sigma=\frac{2}{3}$.
(6) $\frac{2}{3}$.

Questions in Proportion and Progression. Page 77.

(1) 1.
(2) 4.
(3) 69*l.* 11*s.* $6\frac{3}{4}$*d.*
(4) 2, 4, 8.
(5) 14 and 21.
(6) 300.
(7) 10 and 2.
(8) 14 : 9.
(9) 234.
(10) 1, 3, 9.

Binomial Theorem. Page 89.

(1) $(a+x)^8=a^8+8a^7x+28a^6x^2+56a^5x^3+70a^4x^4+56a^3x^5+28a^2x^6+8ax^7+x^8$.

(2) $(a-x)^7=a^7-7a^6x+21a^5x^2-35a^4x^3+35a^3x^4-21a^2x^5+7ax^6-x^7$.

(3) $(x+2y)^7=x^7+14x^6y+84x^5y^2+280x^4y^3+560x^3y^4+672x^2y^5+448xy^6+128y^7$.

(4) $(1+x^2)^{-3}=1-3x^2+6x^4-10x^6+$ &c.

(5) $(a+2x)^{-3}=\frac{1}{a^3}(1-\frac{6x}{a}+\frac{24x^2}{a^2}-\frac{80x^3}{a^3}+$ &c.)

Page 90.

(1) $(a^2+x^2)^{\frac{1}{2}}=a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+$ &c.

Ex. (2) $(a^2-x^2)^{\frac{1}{2}}=a-\frac{x^2}{2a}-\frac{x^4}{8a^3}-\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}-$ &c.

(3) $(a^3-b^3)^{\frac{1}{3}}=a-\frac{b^3}{3a^2}-\frac{b^6}{9a^5}-\frac{5b^9}{81a^8}-$ &c.

(4) $(a^5+x^5)^{\frac{1}{5}}=a+\frac{x^5}{5a^4}-\frac{2x^{10}}{25a^9}+\frac{6x^{15}}{125a^{14}}-$ &c.

(5) $(b^2+x)^{-\frac{1}{2}}=\frac{1}{b}-\frac{x}{2b^3}+\frac{3x^2}{8b^5}-\frac{5x^3}{16b^7}+$ &c.

(6) $(8+1)^{\frac{1}{3}}=2+\frac{1}{3\cdot2^2}-\frac{2}{3\cdot6\cdot2^5}+\frac{5}{3\cdot6\cdot9\cdot2^7}-\frac{5\cdot8}{3\cdot6\cdot9\cdot12\cdot2^{10}}$ &c.

$\therefore \sqrt[3]{9}=2\cdot08008382305.$

COMPOUND INTEREST, ANNUITIES, REVERSIONS, &c. Page 105.

(1) 350*l.* 19*s.* 1¾*d.*
(2) 4 per cent.
(3) 783*l.* 12*s.* 7*d.*
(4) 43*l.* 6*s.* 11¼*d.*
(5) 210*l.* 6*s.*
(6) 214*l.* 17*s.* 5*d.*
(7) 14·206 years.
(8) 20·149 years.
(9) 4 per cent.
(10) 22⅓ years.

Page 106.

(1) 215*l.* 10*s.* 1½*d.*
(2) 988*l.* 11*s.* 2*d.*
(3) 681*l.* 11*s.* 7*d.*
(4) 279*l.* 0*s.* 3*d.*

Page 107.

(1) 2,775*l.* 1*s.* 10*d.*
(2) 551*l.* 15*s.* 3*d.*
(3) $28\frac{4}{7}$ years.
(4) 195*l.* 7*s.* 8*d.*
(5) 258*l.* 10*s.* 6¾*d.*
(6) 46*l.* 5*s.*
(7) 5 per cent.

Page 108.

(1) 175*l.* 1*s.* 1½*d.*
(2) 882*l.* 14*s.* 1½*d.*
(3) 50*l.*

INDETERMINATE COEFFICIENTS. Page 112.

(1) $\frac{n(n+1)}{2}.$
(2) $\frac{n(2n+1)(2n-1)}{3}.$
(3) $\left\{\frac{n(n+1)}{2}\right\}^2.$
(4) $\frac{n^3+3n^2+2n}{3}.$

PILING OF SHOT. Page 115.

Ex.(1) 120. (2) 4960. (3) 140. (4) 1785.
(5) 392. (6) 23405. (7) 3640.

PERMUTATIONS AND COMBINATIONS. Page 118.

Permutations.—(1) 479001600. (2) 720.
(3) 1260. (4) 64.
Combinations (p. 120).—(1) 210. (2) 455.
(3) 15. (4) 65780. (5) 118.
Page 122.—(6) 50063860, 5006386, 424270 times.
(7) 84 times.

EXTRACTION OF THE SQUARE ROOT. Page 124.

(1) $2x^2-6ax-9a^2$. (2) $a^2x-6ax^2-4x^3$.
(3) x^2-2x+3, Rem. $12x$. (4) $3x^3-2x^2+x-4$.

EXTRACTION OF THE CUBE ROOT. Page 126.

(1) $2x+a$. (2) x^2-2x+1. (3) $4x^2-6x-9$.

GREATEST COMMON MEASURE. Page 131.

(1) a^2-x^2. (2) $x+6$. (3) $x-5$. (4) x^2-2x+1.

End of the Answers to the Algebraical Exercises.

ANSWERS TO THE EXERCISES IN THE TRIGONOMETRY.

Right-angled Triangles. Page 145.

Ex.(1) Perp. 384·045 ft., Hyp. 480·036 ft.
(2) 78·4 ft. (3) 118·57 ft. (4) 7,936 miles.
(5) 8,000 miles. (6) 935·76 ft. (7) 56·65 ft.
(8) Perp. 216 ft., Hyp. 270 ft.
(9) *53° 7′ 48″.* (10) Ship 4100 ft., hill 289 ft.

Oblique-angled Triangles. Page 149.

Ex.(1) $a=471\cdot5$, $b=371\cdot9$.

(2) $A=40°\ 33'\ 13''$, $C=61°\ 13'\ 47''$, $a=101\cdot617$.

(3) 296·54 ft. and 306·19 ft. (4) 1426 ft.

(5) 2292·26 yds. and 2298·05 yds.

(6) 45° 13′ 55″, or 134° 46′ 5″.

Page 151.

(1) 19·26.

(2) Side 112·6 yds., angles 57° 28′ and 64° 35′.

(3) $B=37°\ 20'$, $C=115°\ 36'$.

(4) 57·623 ft. (5) $c=15\cdot241$.

Page 158.

(1) $A=132°\ 7'\ 12''$.

(2) $A=27°\ 4'$, $B=37°\ 20'$ $C=115°\ 36'$.

(3) 41° 24′ 35″, 55° 46′ 16″, 82° 49′ 9″.

(4) $B=54°\ 33'\ 25''$.

(5) 301·01 ft., 719·522 ft., 629·101 ft.

Page 178.

(1) $\text{Sin } 9°=\frac{1}{4}\left\{\sqrt{(3+\sqrt{5})}-\sqrt{(5-\sqrt{5})}\right\}$,
$\cos 9°=\frac{1}{4}\left\{\sqrt{(3+\sqrt{5})}+\sqrt{(5-\sqrt{5})}\right\}$.

(2) $\text{Sin } 27°=\frac{1}{4}\left\{\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})}\right\}$,
$\cos 27°=\frac{1}{4}\left\{\sqrt{(5+\sqrt{5})}+\sqrt{(3-\sqrt{5})}\right\}$.

(3) $x=15°$.

(7) $\text{Tan } 67°\ 30' = 1+\sqrt{2}$, $\sin 22°\ 30'=\frac{1}{2}\sqrt{(2-\sqrt{2})}$.

(13) $\text{Sin } A=\pm\frac{4}{5}$, or $\pm\frac{\sqrt{2}}{10}$.

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